Solution Strategy for Single-Phase Governing Equations

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Final Set of Equations

Mass Balance

$$\frac{\partial(A\rho)}{\partial t}+\frac{\partial(\dot{m})}{\partial s}=0$$

Momentum Balance

$$-\frac{\partial p}{\partial s}=\frac{1}{A}\frac{\partial(\dot{m})}{\partial t}+\frac{1}{A}\frac{\partial(\rho A u^2)}{\partial s}+\tau_w\frac{P}{A}+\rho g\frac{\partial H}{\partial s}$$

Energy Balance

$$\rho A\frac{\partial(h)}{\partial t}+\rho A u\frac{\partial(h)}{\partial s}=uA\frac{\partial p}{\partial s}+A\frac{\partial p}{\partial t}+q''_{surface}P-\frac{\partial(q''_{axial}A)}{\partial s}-W'$$

Solution Strategy-I

	independent	dependent	
Variables	A, H, P, t, s	$\rho, u, \dot{m}, p, \tau_w, h, q''_{surface}, q''_{axial}$	
Equations	3	u, p, h	chosen as dependent
Closing relations	5		

$\rho=\rho(p,h)$ $\quad q''_{surface}=q''_{surface}(\text{specified}/u,\text{properties})$

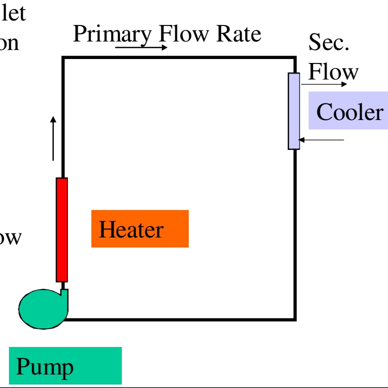
$\dot{m}=\rho A u$ $\quad q''_{axial}=\text{negligible}$

$\tau_w=\tau_w(\rho,\text{geometry},u,\text{viscosity})$

system is mathematically closed

Illustrative Application

- To illustrate the application let us consider forced convection loop
- For a given heater power, assuming large secondary flow in cooler, and a given pump characteristics, estimate, (a) steady mass flow rate circulating, (b) fluid temperature distribution



10:55 5/16

Application-I

Mass Balance

$$\frac{\partial(A\cancel{\rho})}{\cancel{\partial t}}+\frac{\partial(\dot m)}{\partial s}=0 \Rightarrow$$

Mass flow rate is constant along the duct

For incompressible fluid, even under transient, instantaneous mass flow rate does not vary along the length.

10:55 6/16

Application-II

Momentum Balance

Pressure term | Transient term | Acceleration term | Friction term | Gravitation term

$$-\frac{\partial p}{\partial s}=\frac{1}{A}\frac{\partial(\dot m)}{\partial t}+\frac{1}{A}\frac{\partial(\rho A u^2)}{\partial s}+\tau_w\frac{P}{A}+\rho g\frac{\partial H}{\partial s} \quad (11)$$

We shall integrate term by term over the entire loop

Pressure term

$$-\oint_{loop}\frac{\partial p}{\partial s}ds=0$$

Transient term

$$\oint_{loop}\frac{1}{A}\frac{\partial(\dot m)}{\partial t}ds=\frac{d\dot m}{dt}\oint_{loop}\frac{ds}{A}=\frac{d\dot m}{dt}\sum_{all\ links}\frac{L_i}{A_i}$$

10:55 7/16

Application-III

Acceleration term

$$\sum_{all\ links}\int\frac{1}{A}\frac{\partial(\rho A u^2)}{\partial s}ds=\sum_{all\ links}\int\frac{\partial\left(\dot m^2/\rho A^2\right)}{\partial s}ds$$

$$=\frac{\dot m^2}{\rho}\sum_{all\ links}\left(\frac{1}{A_{i+1}^2}-\frac{1}{A_i^2}\right)=0$$

Friction term

$$\oint_{loop}\tau_w\frac{P}{A}ds=\oint_{loop}\frac{\rho u^2}{2}f\frac{4}{d_{hyd}}ds \quad =\frac{4\dot m^2}{2\rho}\sum_{i=all\ links}\frac{f_iL_i}{d_{hyd-i}}\frac{1}{A_i^2}$$

$$where,\ f=16(\mathrm{Re})^{-1} \quad \forall\,\mathrm{Re}<1189.4$$

$$=0.079(\mathrm{Re})^{-0.25}\ \forall\,\mathrm{Re}>1189.4$$

By adding minor losses, we can write

$$=\frac{4\dot m^2}{2\rho}\sum_{i=all\ links}\frac{1}{A_i^2}\left(\frac{f_iL_i}{d_{hyd-i}}+\frac{K_i}{4}\right)$$

10:55 8/16

Application-IV

Gravitation term

$$\oint_{loop}\rho g\frac{\partial H}{\partial s}ds=\rho g\sum_{all\ links}\left(H_{exit-i}-H_{inlet-i}\right)=0$$

For pump link

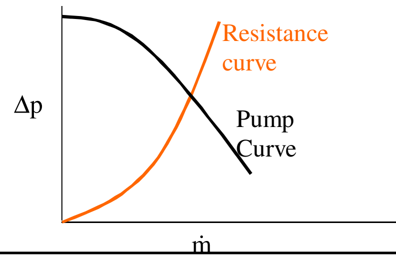
$$-\Delta p_{pump}=-\rho g H_{pump}$$

Equating pressure term to sum of all the other terms, we get

$$\frac{d\dot m}{dt}\sum_{all\ links}\frac{L_i}{A_i}+\frac{4\dot m^2}{2\rho}\sum_{i=all\ links}\frac{1}{A_i^2}\left(\frac{f_iL_i}{d_{hyd-i}}+\frac{K_i}{4}\right)-\rho g H_{pump}=0$$

Application-V

- The integrated momentum equation has only one unknown viz., $\dot{m}$
- For a given initial condition, the solution can be marched in time. Any standard procedure for solving ODE can be employed
- For steady situations, the First term drops out and any standard procedure for solving non-linear equations can be employed



Application-VI

Energy Balance

$$\rho A\frac{\partial(h)}{\partial t}+\rho A u\frac{\partial(h)}{\partial s}=uA\frac{\partial p}{\partial s}+\cancel{A\frac{\partial p}{\partial t}}+q''_{surface}P-\cancel{\frac{\partial(q''_{axial}A)}{\partial s}}-W'+\tau_w Pu$$

For steady case

$$\cancel{\rho A\frac{\partial(h)}{\partial t}}+\rho A u\frac{\partial(h)}{\partial s}=\cancel{uA\frac{\partial p}{\partial s}}+q''_{surface}P-W'+\cancel{\tau_w Pu}$$

small

For non-pump links

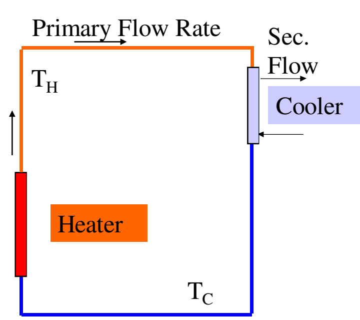
$$\dot{m}c_p\frac{dT}{ds}=q''_{surface}P$$

Linear variation for constant heat flux case and exponential for constant U and T_{amb} case.

Natural Circulation-I

The Flow is driven by density variations and is also called Gravity Driven Flow or Thermosyphon Flow.

In these flows, momentum and energy are coupled and so need to be simultaneously solved.



Assumptions

Density only a function of Temperature. Usually assumed linear.

$$\rho=\rho_R(1-\beta(T-T_R))$$

Natural Circulation-II

Boussinesq Approximation is valid. This implies that density variation with temperature need to be accounted for only in body force term of the momentum equation.

Transport properties (μ, c_p, k) are constant.

Mass Balance

$$\cancel{\frac{\partial(A\rho)}{\partial t}}+\frac{\partial(\dot{m})}{\partial s}=0$$

Mass flow rate is constant along the duct

10:55

Natural Circulation-III

13/16

Momentum Balance

The only difference is the integration of the gravitation term.

Gravitation term

$$\oint_{loop} \rho g \frac{\partial H}{\partial s} ds = \oint_{loop} \rho_R (1-\beta(T-T_R)) g dH$$

$$= \oint_{loop} \rho_R g(1+\beta T_R) dH - \oint_{loop} \rho_R g \beta T dH$$

Thus, the integrated momentum equation leads to

10:55

Natural Circulation-IV

14/16

$$\frac{d\dot{m}}{dt} \sum_{\text{all links}} \frac{L_i}{A_i} + \frac{4\dot{m}^2}{2\rho_R} \sum_{\text{i=all links}} \frac{1}{A_i^2}\left(\frac{f_i L_i}{d_{hyd-i}} + \frac{K_i}{4}\right) - \oint_{loop} \rho_R g \beta T dH = 0$$

This equation cannot be solved unless the temperature distribution is obtained. Let us consider only steady state.

Energy Balance

$$\dot{m} c_p \frac{dT}{ds} = q''_{surface} P$$

In heater $\quad T = T_C + \frac{q''_{surface} P}{\dot{m} c_p} s$

At $s = L_H$ $\quad \Rightarrow T_H = T_C + \frac{q''_{surface} P}{\dot{m} c_p} L_H \quad \Rightarrow T_C = T_H - \frac{q''_{surface} P}{\dot{m} c_p} L_H$

10:55

Natural Circulation-V

15/16

Post heater $\quad T = T_H$

In cooler $\quad \dot{m} c_p \frac{dT}{ds} = U(T - T_\infty) P$

Integration leads to

$$\frac{T - T_\infty}{T_H - T_\infty} = e^{-\frac{UPs}{\dot{m} c_p}}$$

At $s = L_C$ $\quad \frac{T_C - T_\infty}{T_H - T_\infty} = e^{-\frac{UPL_C}{\dot{m} c_p}} \quad \Rightarrow T_C = T_\infty + (T_H - T_\infty) e^{-\frac{UPL_C}{\dot{m} c_p}}$

Post cooler $\quad T = T_C$

Elimination of T_C from the two relation leads to

10:55

Natural Circulation-VI

16/16

$$(T_H - T_\infty) = \frac{q'' P_H L_H}{\dot{m} c_p} \frac{1}{\left(1 - e^{-\frac{UP_C L_C}{\dot{m} c_p}}\right)}$$

Solution Procedure

1. Assume $\dot{m}$
2. Compute temperature profiles
3. Check if integrated momentum equation is satisfactory
4. Repeat till convergence