

Transient Analysis Using MoC

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Agenda

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- We now shift to case where the equations cannot be reduced to an ODE
- First we shall look at single-phase systems and then turn to homogeneous two-phase flow.
- In both cases we shall obtain solutions using the method of characteristics
- This method is commonly used for hyperbolic equations

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MoC-I

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- Consider the equation of the type

$$A \frac{\partial(\phi)}{\partial t} + B \frac{\partial(\phi)}{\partial s} = C$$

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- In general, the solution of ϕ can be continuous or discontinuous
- Note that discontinuity would imply multiple values of derivatives
- Continuity would imply unique values of derivatives

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MoC-II

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- If the function is continuous, then chain rule will be valid, implying

$$d\phi = \frac{\partial(\phi)}{\partial t} dt + \frac{\partial(\phi)}{\partial s} ds$$

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- We can view equations (1) and (2) as two equations dictating the nature of derivatives

$$\begin{bmatrix} dt & ds \\ A & B \end{bmatrix} \begin{Bmatrix} \frac{\partial(\phi)}{\partial t} \\ \frac{\partial(\phi)}{\partial s} \end{Bmatrix} = \begin{Bmatrix} d\phi \\ C \end{Bmatrix}$$

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MoC-III

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- The necessary condition for continuity is that the determinant of the coefficient matrix cannot be zero

- For a discontinuity to exist,

$$\begin{vmatrix} dt & ds \\ A & B \end{vmatrix} = 0$$

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Or $\frac{ds}{dt} = \frac{B}{A}$

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- The locus of the line obtained by integration of Eq.(5) is called the characteristic line

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MoC-IV

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- If B and A are real, then the characteristic direction is real and the governing equation will be classified as hyperbolic.
- The simplified energy equation valid at low speed flows,

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial s} = \frac{q'' P}{\rho c_p A}$$

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Would classify as hyperbolic as $ds/dt = u$

- We shall now see as to how to get the solution of the above equation

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MoC-V

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- As we have already seen the method to get the velocity previously. The aim now will be to obtain the temperature distribution from the energy equation
- As stated in the previous slide, the characteristic direction is given by

$$\frac{dt}{ds} = \frac{1}{u}$$

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- For single-phase constant density flow u is constant along a constant diameter duct, but can change with time

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MoC-VI

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- Let us consider the case for illustration for the case in which velocity exponentially decays with time following the expression,

$$u = e^{-t}$$

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- Along the characteristic line Eq. (6) can be written as

$$\frac{dT}{dt} = \frac{q'' P}{\rho c_p A}$$

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Note that we have applied chain rule. The discontinuities occur across characteristic and not along them

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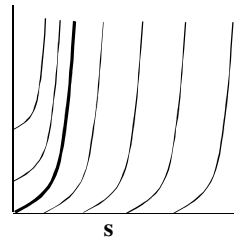
MoC-VII

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- Integration of the characteristic equation sketches the characteristic path. In this case it turns out to be the path line

$$\int_{s_0}^s ds = \int_{t_0}^t u dt = \int_{t_0}^t e^{-t} dt = e^{-t_0} - e^{-t} = s - s_0 \quad (10)$$

- We have sketched the loci in the figure shown. The s_0 and t_0 are the initial location and time, while s and t are any arbitrary space and time



MoC-VIII

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- For obtaining the solution, we need to know whether the particle was inside or outside the domain when the transient started
- The characteristic originating from the origin decides that. All that are on right of it were within and those to the left came later.
- For any given s, t , we find s_0 for $t_0 = 0$. This is done by using Eq. (10).

$$s_0 = s - (1 - e^{-t}) \quad (11)$$

- If $s_0 > 0$, then the characteristic originates from right. However, if $s_0 < 0$, then it originates from left

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MoC-IX

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- To obtain the solution, we need to specify the initial and boundary conditions for the governing equation (Eq. (6))
- The initial condition in general can be written as

$$T(s, 0) = T_o(s)$$

- The boundary condition can be written as

$$T(0, t) = T_b(t)$$

- The solution for the points that are to the right of characteristic from origin is

$$T(s, t) = T_o(s_0) + \int_0^t \frac{q'' P}{\rho c_p A} dt \quad (12)$$

MoC-X

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- For points that are on the left, we shall proceed as follows
- The time t_0 for these are not zero and needs to be found out. This is found from Eq. (10) by substituting $s_0 = 0$

$$e^{-t_0} = s + e^{-t} \Rightarrow t_0 = \ln\left(\frac{1}{s + e^{-t}}\right) \quad (13)$$

- The solution for T can now be found from

$$T(s, t) = T_b(t_0) + \int_{t_0}^t \frac{q'' P}{\rho c_p A} dt \quad (14)$$

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Solution for Homogeneous Flow

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- We shall restrict to the homogeneous model
- Here again, if we invoke incompressibility of each phase, very simple and elegant equations are obtained
- These are valid for cases in which pressure is high, but pressure changes are slow such as for station transients
- The final equation is amenable to be solved by MoC

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Governing Equations-I

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- The mass balance can be expanded as

$$\frac{\partial \rho_m}{\partial t} + u_m \frac{\partial \rho_m}{\partial s} + \rho_m \frac{\partial u_m}{\partial s} = 0 \quad (15)$$

- The energy balance can be written as

$$\frac{\partial h_m}{\partial t} + u_m \frac{\partial h_m}{\partial s} = \frac{q'_w}{A \rho_m} \quad (16)$$

- To simplify algebra, we can introduce the substantial derivative and rewrite the above two equations as

$$\frac{D \rho_m}{Dt} = -\rho_m \frac{\partial u_m}{\partial s} \quad (17) \quad \frac{D h_m}{Dt} = \frac{q'_w}{A \rho_m} \quad (18)$$

- The further analysis shall assume that ρ_f , ρ_g , h_f and h_g are assumed constants,

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Governing Equations-II

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- Eq. (17) can now be rewritten in terms of specific volume as

$$-\frac{1}{v_m^2} \frac{D v_m}{Dt} = -\rho_m \frac{\partial u_m}{\partial s} \Rightarrow \frac{D v_m}{Dt} = v_m \frac{\partial u_m}{\partial s} \\ \Rightarrow v_{fg} \frac{D x}{Dt} = v_m \frac{\partial u_m}{\partial s} \Rightarrow \frac{D x}{Dt} = \frac{v_m}{v_{fg}} \frac{\partial u_m}{\partial s} \quad (19)$$

- Eq. (18) can be rewritten as

$$h_{fg} \frac{D x}{Dt} = \frac{q'_w}{A \rho_m} \Rightarrow \frac{D x}{Dt} = \frac{q'_w}{h_{fg} A \rho_m} \quad (20)$$

- Eqs. (19) and (20) leads to $\frac{\partial u_m}{\partial s} = \frac{v_{fg} q'_w}{h_{fg} A} \quad (21)$

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Governing Equations-III

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- Eq. (21) leads to an important conclusion that the instantaneous variation of the homogeneous velocity is dictated by linear heat rate just as in steady flow.
- If linear heat rate is constant, then the velocity variation shall be linear
- Restricting the case to constant linear heat rate for simplicity, we can write

$$u_m = a s + b$$

- The characteristic direction shall be given by

$$\frac{ds}{dt} = a s + b \Rightarrow \frac{s+b}{s_0+b} = e^{at} - e^{-at_0} \quad (22)$$

- Rest of the concepts are similar. We need to divide the region on either side of the characteristic from the origin following the same procedure
- Similarly, the energy equation can be integrated along the characteristic to obtain the value of x , for known x_0

$$\Rightarrow \frac{dx}{dt} = \frac{q'_w (v_f + x v_{fg})}{h_{fg} A} \quad (23)$$

$$\Rightarrow \frac{1}{v_{fg}} \ln \left(\frac{v_f + x v_{fg}}{v_f + x_0 (s_0, t_0) v_{fg}} \right) = \frac{q'_w}{h_{fg} A} (t - t_0) \quad (24)$$

- The value of s_0 and t_0 are obtained very similarly by manipulating the characteristic equation