Transient Analysis Using MoC

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Recap-I

- We have already studied one dimensional area averaged governing equations for mass, momentum and energy for fluid flow in heated systems
- We also restricted ourselves to steady systems most of the times as the interest was to get solutions by simple means either analytical/numerical

Recap-II

- Limited solutions for the transient cases where the equations could be reduced to an ODE
- This was for incompressible single phase flow where, the velocity was constant along the ducts and the equations could be integrated over space and governing equations could be simplified
- An applied case of loss of flow could also be analysed this way

Agenda

- We now shift to case where the equations cannot be reduced to an ODE
- First we shall look at single-phase systems and then turn to homogeneous two-phase flow.
- In both cases we shall obtain solutions using the method of characteristics
- This method is commonly used for hyperbolic equations

MoC-I

- Consider the equation of the type

$$A\frac{\partial(\phi)}{\partial t}+B\frac{\partial(\phi)}{\partial s}=C \qquad (1)$$

- In general, the solution of ϕ can be continuous or discontinuous
- Note that discontinuity would imply multiple values of derivatives
- Continuity would imply unique values of derivatives

MoC-II

- If the function is continuous, then chain rule will be valid, implying

$$d\phi=\frac{\partial(\phi)}{\partial t}dt+\frac{\partial(\phi)}{\partial s}ds \qquad (2)$$

- We can view equations (1) and (2) as two equations dictating the nature of derivatives

$$\begin{bmatrix} dt & ds \\ A & B \end{bmatrix}\begin{Bmatrix} \frac{\partial(\phi)}{\partial t} \\ \frac{\partial(\phi)}{\partial s} \end{Bmatrix}=\begin{Bmatrix} d\phi \\ C \end{Bmatrix} \qquad (3)$$

MoC-III

- The necessary condition for continuity is that the determinant of the coefficient matrix cannot be zero
- For a discontinuity to exist,

$$\begin{vmatrix} dt & ds \\ A & B \end{vmatrix}=0 \qquad (4)$$

Or $$\frac{ds}{dt}=\frac{B}{A} \qquad (5)$$

- The locus of the line obtained by integration of Eq.(5) is called the characteristic line

MoC-IV

- If B and A are real, then the characteristic direction is real and the governing equation will be classified as hyperbolic.
- The simplified energy equation valid at low speed flows,

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial s}=\frac{q''P}{\rho c_p A} \qquad (6)$$

Would classify as hyperbolic as ds/dt = u

- We shall now see as to how to get the solution of the above equation

MoC-V

- As we have already seen the method to get the velocity previously (say loss flow accident), The aim now will be to obtain the temperature distribution from the energy equation
- As stated in the previous slide, the characteristic direction is given by

$$\frac{dt}{ds}=\frac{1}{u} \tag{7}$$

- For single-phase constant density flow u is constant along a constant diameter duct, but can change with time

MoC-VI

- Let us consider the case for illustration for the case in which velocity exponentially decays with time following the expression,

$$u=e^{-t} \tag{8}$$

- Along the characteristic line Eq. (6) can be written as

$$\frac{dT}{dt}=\frac{q''P}{\rho c_p A} \tag{9}$$

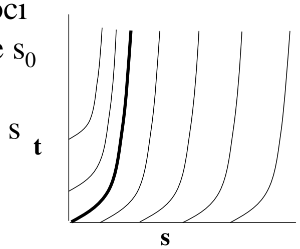
Note that we have applied chain rule. The discontinuities occur across characteristic and not along them

MoC-VII

- Integration of the characteristic equation sketches the characteristic path. In this case it turns out to be the path line

$$\int_{s_0}^{s} ds=\int_{t_0}^{t} u dt=\int_{t_0}^{t} e^{-t} dt=e^{-t_0}-e^{-t}=s-s_0 \tag{10}$$

- We have sketched the loci in the figure shown. The s_0 and t_0 are the initial location and time, while s and t are any arbitrary space and time



MoC-VIII

- For obtaining the solution, we need to know whether the particle was inside or outside the domain when the transient started
- The characteristic originating from the origin decides that. All that are on right of it were within and those to the left came later.
- For any given s,t, we find s_0 for $t_0 = 0$. This is done by using Eq. (10).

$$s_0=s-(1-e^{-t}) \tag{11}$$

- If $s_0 > 0$, then the characteristic originates from right. However, if s < 0, then it originates from left

MoC-IX

- To obtain the solution, we need to specify the initial and boundary conditions for the governing equation (Eq. (6))
- The initial condition in general can be written as

$$T(s,0) = T_o(s)$$

- The boundary condition can be written as

$$T(0,t) = T_B(t)$$

- The solution for the points that are to the right of characteristic from origin is

$$T(s,t) = T_0(s_0) + \int_0^t \frac{q''P}{\rho c_p A} dt \quad (12)$$

MoC-X

- For points that are on the left, we shall proceed as follows
- The time t_0 for these are not zero and needs to be found out. This is found from Eq. (10) by substituting $s_0 = 0$

$$e^{-t_0} = s + e^{-t} \Rightarrow t_0 = \ln\left(\frac{1}{s + e^{-t}}\right) \quad (13)$$

- The solution for T can now be found from

$$T(s,t) = T_B(t_0) + \int_{t_0}^t \frac{q''P}{\rho c_p A} dt \quad (14)$$

Solution for Homogeneous Flow

- We shall restrict to the homogeneous model
- Here again, if we invoke incompressibility of each phase, very simple and elegant equations are obtained
- These are valid for cases in which pressure is high, but pressure changes are slow such as for station transients
- The final equation is amenable to be solved by MoC

Governing Equations-I

- The mass balance can be expanded as

$$\frac{\partial \rho_m}{\partial t} + u_m \frac{\partial \rho_m}{\partial s} + \rho_m \frac{\partial u_m}{\partial s} = 0 \quad (15)$$

- The energy balance can be written as

$$\frac{\partial h_m}{\partial t} + u_m \frac{\partial h_m}{\partial s} = \frac{q'_W}{A\rho_m} \quad (16)$$

- To simplify algebra, we can introduce the substantial derivative and rewrite the above two equations as

$$\frac{D\rho_m}{Dt} = -\rho_m \frac{\partial u_m}{\partial s} \quad (17) \qquad \frac{Dh_m}{Dt} = \frac{q'_W}{A\rho_m} \quad (18)$$

- The further analysis shall assume that ρ_f, ρ_g, h_f and h_g are assumed constants,

Governing Equations-II

- Eq. (17) can now be rewritten in terms of specific volume as

$$-\frac{1}{v_m^{\;2}}\frac{Dv_m}{Dt} = -\rho_m \frac{\partial u_m}{\partial s} \qquad \Rightarrow \frac{Dv_m}{Dt} = v_m \frac{\partial u_m}{\partial s}$$

$$\Rightarrow v_{fg}\frac{Dx}{Dt} = v_m \frac{\partial u_m}{\partial s} \qquad \Rightarrow \frac{Dx}{Dt} = \frac{v_m}{v_{fg}}\frac{\partial u_m}{\partial s} \qquad (19)$$

- Eq. (18) can be rewritten as

$$h_{fg}\frac{Dx}{Dt} = \frac{q_W'}{A\rho_m} \qquad \Rightarrow \frac{Dx}{Dt} = \frac{q_W'}{h_{fg}A\rho_m} \qquad (20)$$

- Eqs. (19) and (20) leads to $\dfrac{\partial u_m}{\partial s} = \dfrac{v_{fg}q_w'}{h_{fg}A}$ (21)

Governing Equations-III

- Eq. (21) leads to an important conclusion that the instantaneous variation of the homogeneous velocity is dictated by linear heat rate just as in steady flow.
- If linear heat rate is constant, then the velocity variation shall be linear
- Restricting the case to constant linear heat rate for simplicity, we can write

$$u_m = a\ s + b$$

- The characteristic direction shall be given by

$$\frac{ds}{dt} = a\ s + b \ \Rightarrow \frac{as+b}{as_0+b} = e^{at} - e^{at_0} \qquad (22)$$

Governing Equations-IV

- Rest of the concepts are similar. We need to divide the region on either side of the characteristic from the origin following the same procedure
- Similarly, the energy equation can be integrated along the characteristic to give

$$\Rightarrow \frac{dx}{dt} = \frac{q_W'\left(v_f + xv_{fg}\right)}{h_{fg}A} \qquad (23)$$

$$\Rightarrow \ln\left(\frac{v_f + xv_{fg}}{v_f + x_0 v_{fg}}\right) = \frac{v_{fg}q_W'}{h_{fg}A}(t - t_0) \qquad (24)$$

- The value of s_0 and t_0 are obtained very similarly by manipulating the characteristic equation