Governing Equations for Single-phase Flows Lecture-2

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ONE DIMENSIONAL ANALYSIS

- In reactor thermal hydraulics one dimensional area averaged approach is most popular.
- While detailed averages can be taken from the three dimensional Navier-Stokes equations, the same can be better understood through the plug flow model approach

Conservation of Mass - I

A – Area (m^2)

u – Velocity (m/s)

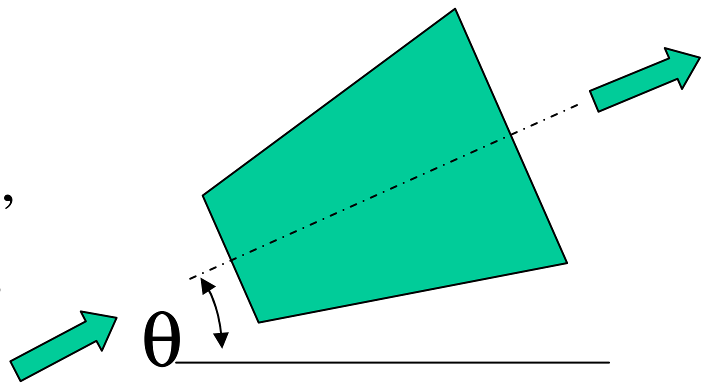
ρ – Density (kg/m^3)

$\dot{m}$ – Mass flow rate (kg/s)

s – Coordinate along pipe (m)

Γ' – Mass source/sink (kg/s-m)

A, u, ρ, $\dot{m}$



$$A + \frac{\partial A}{\partial s}\Delta s$$

$$u + \frac{\partial u}{\partial s}\Delta s$$

$$\rho + \frac{\partial \rho}{\partial s}\Delta s$$

$$\dot{m} + \frac{\partial \dot{m}}{\partial s}\Delta s$$

Rate of accumulation of mass in CV = Mass flow rate into CV – Mass flow rate out of CV +/- Rate of Mass generated / destructed in CV

Conservation of Mass - II

$$\frac{\partial(\rho A \Delta s)}{\partial t} = \cancel{\dot{m}} - \left(\cancel{\dot{m}} + \frac{\partial(\dot{m}_g)}{\partial s}\Delta s \right) \pm \Gamma' \Delta s$$

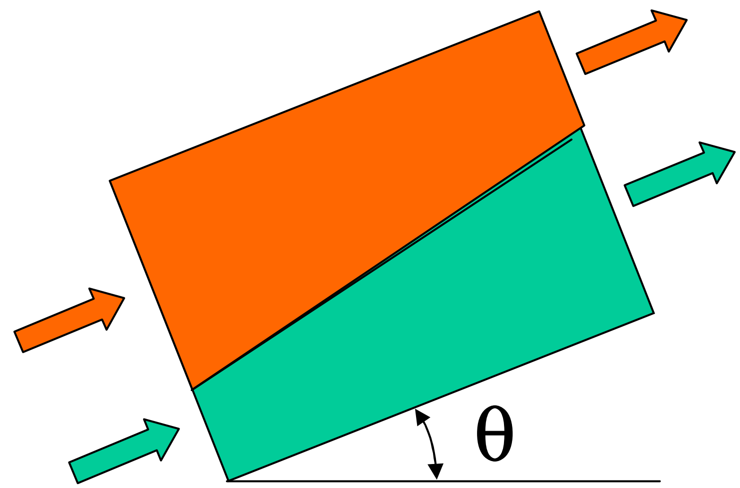
$$\boxed{\frac{\partial(\rho A)}{\partial t} + \left(\frac{\partial(\dot{m}_g)}{\partial s} \right) = \pm \Gamma'} \qquad 1$$

The control volume need not extend over the entire cross section. An example is given in next slide

Conservation of Mass - III

Gas $$\frac{\partial(\rho_g A_g)}{\partial t} + \frac{\partial(\dot{m}_g)}{\partial s} = \Gamma_g'$$

liquid $$\frac{\partial(\rho_l A_l)}{\partial t} + \frac{\partial(\dot{m}_l)}{\partial s} = \Gamma_l'$$



Note: In boiling flows $\Gamma_g' = -\Gamma_l'$

Conservation of Momentum - I

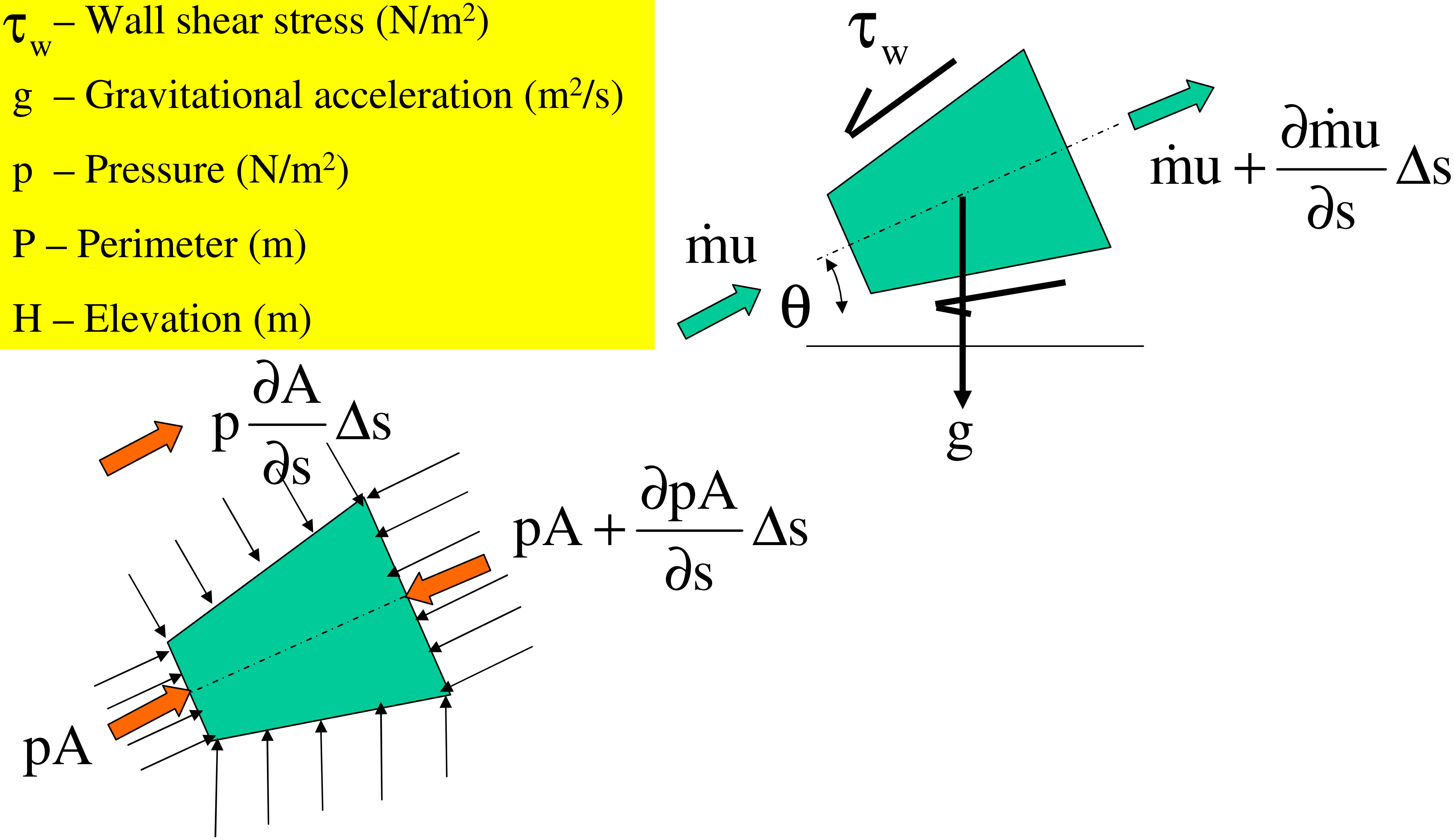
τ_w – Wall shear stress (N/m^2)

g – Gravitational acceleration (m^2/s)

p – Pressure (N/m^2)

P – Perimeter (m)

H – Elevation (m)



Conservation of Momentum - II

Rate of accumulation of momentum in CV = Momentum rate into CV − Momentum rate out of CV + Sum of all forces in positive direction acting on CV

1 2 3 4

Term-1 $\frac{\partial(\rho A \Delta s u)}{\partial t}$

Term-2 $\dot{m}u$

Term-3 $\dot{m}u + \frac{\partial \dot{m}u}{\partial s}\Delta s$

Conservation of Momentum - III

Term-4 = Pressure force + Shear force + Gravity force

Pressure force $= pA - \left(pA + \frac{\partial pA}{\partial s}\Delta s \right) + p\frac{\partial A}{\partial s}\Delta s$

$= -A\frac{\partial p}{\partial s}\Delta s$

Shear force $= -\tau_w P \Delta s$

Gravity force $= -\rho A \Delta s g \mathrm{Sin}(\theta)$

Substitution of all the terms lead to

Conservation of Momentum - IV

$$\frac{\partial(\rho A u \Delta s)}{\partial t} = \cancel{\dot{m}u} - \left(\cancel{\dot{m}u} + \frac{\partial(\dot{m}u)}{\partial s} \right) - A\frac{\partial p}{\partial s}\Delta s - \tau_w P \Delta s - \rho A \Delta s g \mathrm{Sin}\theta$$

$$\Rightarrow \quad \frac{\partial(\rho A u)}{\partial t} + \frac{\partial(\rho A u^2)}{\partial s} = -A\frac{\partial p}{\partial s} - \tau_w P - \rho A g \mathrm{Sin}\theta \qquad 2$$

$$\Rightarrow \quad \frac{\partial(\dot{m})}{\partial t} = -\frac{\partial(\dot{m}u)}{\partial s} - A\frac{\partial p}{\partial s} - \tau_w P - \rho A g \mathrm{Sin}\theta \qquad 3$$

Eqs. (2) and (3) are the conservative forms of the momentum equation

Conservation of Momentum - V

- We can get non-conservative form by expanding the LHS and using mass conservation equation

$$\rho A\frac{\partial u}{\partial t}+\cancel{u\frac{\partial \rho A}{\partial t}}+\rho Au\frac{\partial u}{\partial s}+\cancel{u\frac{\partial \rho Au}{\partial s}}=-A\frac{\partial p}{\partial s}-\tau_w P-\rho AgSin\theta$$

$$\Rightarrow \quad \rho A\frac{\partial u}{\partial t}+\rho Au\frac{\partial u}{\partial s}=-A\frac{\partial p}{\partial s}-\tau_w P-\rho AgSin\theta \qquad (4)$$

The above expression assumes source and sink of mass is zero.

Mechanical Energy Equation - I

- Multiplication of momentum equation with velocity will give mechanical energy equation

$$\rho A u \frac{\partial u}{\partial t} + \rho A u^2 \frac{\partial u}{\partial s} = -uA\frac{\partial p}{\partial s} - u\tau_w P - u\rho A g \sin\theta \qquad 5$$

$$\rho A \frac{\partial \, u^2/2}{\partial t} + \rho A u \frac{\partial \, u^2/2}{\partial s} = -uA\frac{\partial p}{\partial s} - u\tau_w P - u\rho A g \frac{\partial H}{\partial s} \qquad 6$$

For steady, inviscid and incompressible flow, we get

$$Au\frac{\partial \, \rho u^2/2}{\partial s} + Au\frac{\partial p}{\partial s} + uA\frac{\partial(\rho g H)}{\partial s} = 0$$

Mechanical Energy Equation - I

$$\Rightarrow \quad \frac{\partial\left(p+\rho u^2/2+\rho g H\right)}{\partial s}=0$$

$$\Rightarrow \quad p+\rho u^2/2+\rho g H=\text{Constant}$$

Bernoulli's Equation

I-Law of Thermodynamics

h - Specific enthalpy (J/kg)

Definitions

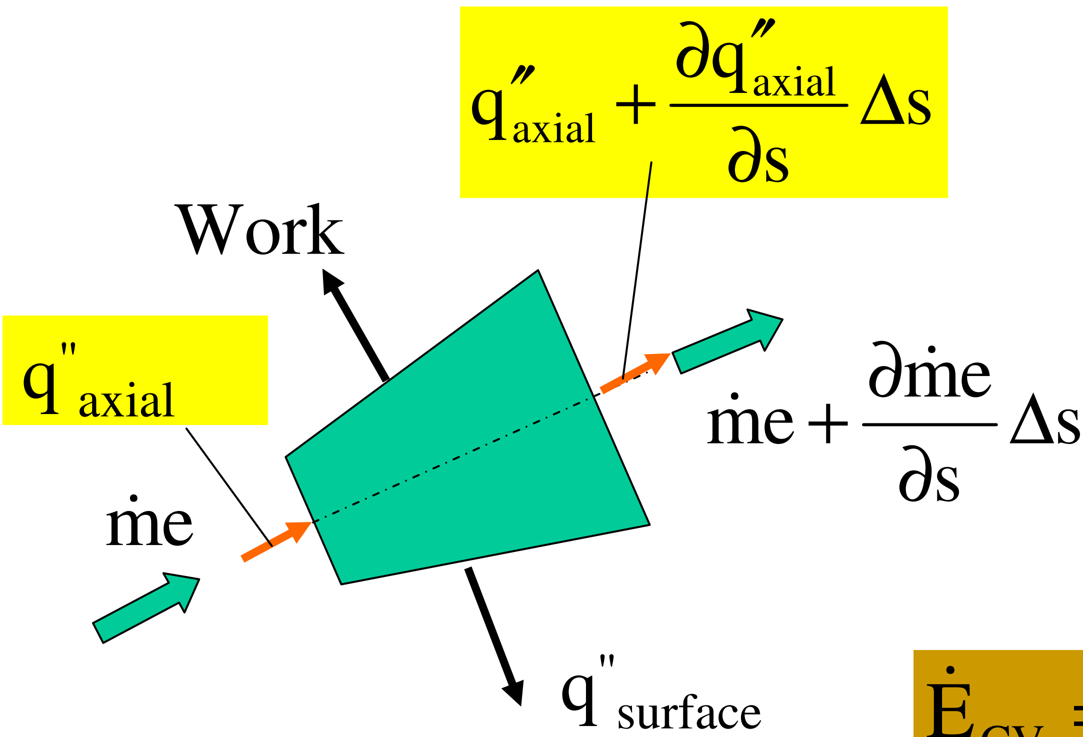
Specific Flow Energy

$$e = h + \frac{u^2}{2} + gH$$

Specific Energy

$$i = \underbrace{h - \frac{p}{\rho}}_{\text{Internal energy}} + \frac{u^2}{2} + gZ$$

I-Law - II



$$\dot{E}_{CV} = \dot{Q} - \dot{W} + \dot{m}\left(e_{inlet} - e_{exit}\right)$$

$$q''_{axial} = -k\frac{\partial T}{\partial s}$$

$$q''_{surface} = U(T - T_\infty)$$

k – Thermal Conductivity (W/m-K)

U – Overall heat transfer coefficient (W/m^2-K)

I-Law - III

$$\dot{E}_{CV} = \frac{\partial\left(A\Delta s\rho i\right)}{\partial t} = A\Delta s \frac{\partial\left(\rho\left[e - {}^{p}\!/\!_{\rho}\right]\right)}{\partial t}$$

- Using Taylor series, we can write

$$\dot{m}_{inlet}e_{inlet} - \dot{m}_{inlet}e_{exit} = \dot{m}e - \left(\dot{m}e + \frac{\partial(\dot{m}e)}{\partial s}\Delta s\right)$$

$$= -\frac{\partial(\dot{m}e)}{\partial s}\Delta s$$

I-Law - IV

- We can write

$$\dot{W} = \tau_w P \Delta s u_H + W'_{CV} \Delta s$$

$$\dot{Q} = q''_{surface} P \Delta s + q''_{axial} A - \left(q''_{axial} A + \frac{\partial \left(q''_{axial} A \right)}{\partial s} \Delta s \right)$$

$$= q''_{surface} P \Delta s - \frac{\partial \left(q''_{axial} A \right)}{\partial s} \Delta s$$

Substitution of all the terms lead to

$$\frac{\partial(\rho A e)}{\partial t} + \frac{\partial(\rho A u e)}{\partial s} = A \frac{\partial p}{\partial t} + q''_{surface} P - \tau_w P u - \frac{\partial \left(q''_{axial} A \right)}{\partial s} - W'_{CV}$$

Conservative form

I-Law - V

- Using mass conservation we can write

$$\rho A \frac{\partial e}{\partial t} + \rho A u \frac{\partial e}{\partial s} = A \frac{\partial p}{\partial t} + q'' P - \tau_w P u - \frac{\partial \left(q''_{axial} A \right)}{\partial s} - W'_{CV}$$

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- Substituting the expression for e, we get

$$\rho A \frac{\partial \left(h + {u^2}/{2} + gH \right)}{\partial t} + \rho A u \frac{\partial \left(h + {u^2}/{2} + gH \right)}{\partial s} =$$

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$$A \frac{\partial p}{\partial t} + q''_{surace} P - \tau_w P u - \frac{\partial \left(q''_{axial} A \right)}{\partial s} - W'_{CV}$$

I-Law - VI

- Subtracting both sides of mechanical energy equation Eq. (6) from I–Law Eq. (9) , we can write the thermal energy equation:

$$\rho A \frac{\partial(h)}{\partial t} + \rho A u \frac{\partial(h)}{\partial s} = u A \frac{\partial p}{\partial s} + A \frac{\partial p}{\partial t} + q''_{surface} P - \frac{\partial\left(q''_{axial} A\right)}{\partial s} - W' \qquad (10)$$

- Here we have assumed that shear work is lost out to the surroundings. For insulated systems this gets ploughed back and hence we have to add $\tau_w P u$ on RHS. If there is energy split, this has to be suitably handled