ME 209
Basic Thermodynamics
Introduction to Compressible Flow-II

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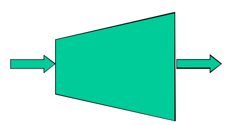
ONE DIMENSIONAL ANALYSIS

- Let us now try to apply mass momentum and energy for a one dimensional flow
- The approach is very popular during design phase for many practical applications.
- Let us develop these from the concepts developed thus far

Conservation of Mass - I

A – Area (m^2)
V – Velocity (m/s)
ρ – Density (kg/m^3)
$\dot{m}$ – Mass flow rate (kg/s)
x – Coordinate along pipe (m)

A, V, ρ, $\dot{m}$



$A + \frac{\partial A}{\partial x}\Delta x$

$V + \frac{\partial V}{\partial x}\Delta x$

$\rho + \frac{\partial \rho}{\partial x}\Delta x$

$\dot{m} + \frac{\partial \dot{m}}{\partial x}\Delta x$

Rate of accumulation of mass in CV = Mass flow rate into CV − Mass flow rate out of CV

$$\frac{\partial(\rho A \Delta x)}{\partial t} = \dot{m} - \left(\dot{m} + \frac{\partial \dot{m}}{\partial x}\Delta x\right) \qquad \Rightarrow \frac{\partial(\rho A)}{\partial t} + \left(\frac{\partial \dot{m}}{\partial x}\right) = 0$$

Conservation of Mass - I

- At steady state

$$\frac{\partial(\rho A)}{\partial t} + \left(\frac{\partial \dot{m}}{\partial x}\right) = 0 \qquad \Rightarrow \dot{m} = \rho A V = \text{Constant}$$

$$\Rightarrow \rho A dV + AV d\rho + V\rho dA = 0 \qquad (1)$$

$$\Rightarrow \frac{dV}{V} = -\frac{dA}{A} - \frac{d\rho}{\rho} \qquad (2)$$

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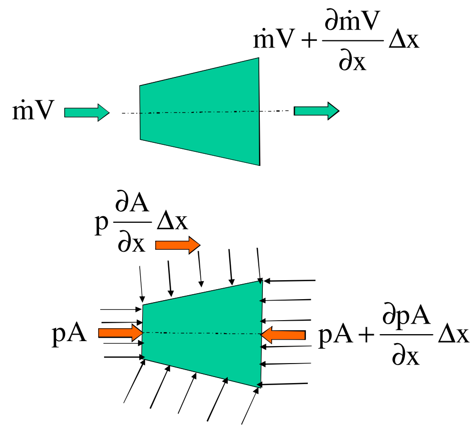
Conservation of Momentum - I



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Conservation of Momentum - II

Rate of accumulation of momentum in CV	=	Momentum rate into CV	–	Momentum rate out of CV	+	Sum of all forces in positive direction acting on CV
$\rho A \Delta x V$		$\dot{m}V$		$\dot{m}V + \frac{\partial \dot{m}V}{\partial x}\Delta x$		

$$pA - \left(pA + \frac{\partial pA}{\partial x}\Delta x\right) + p\frac{\partial A}{\partial x}\Delta x = -A\frac{\partial p}{\partial x}\Delta x$$

$$\frac{\partial(\rho A u \Delta x)}{\partial t} = \dot{m}V - \left(\dot{m}V + \frac{\partial(\dot{m}V)}{\partial x}\right) - A\frac{\partial p}{\partial x}\Delta x$$

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Conservation of Momentum - IV

- At steady state

$$0 = \frac{d(\dot{m}V)}{dx} + A\frac{dp}{dx}$$

$$\Rightarrow 0 = \frac{d\rho A V^2}{dx} + A\frac{dp}{dx}$$ **As mass flow rate is constant**

$$\Rightarrow 0 = \rho A V\frac{dV}{dx} + A\frac{\partial p}{\partial x}$$

$$\Rightarrow 0 = V\frac{dV}{dx} + \frac{1}{\rho}\frac{\partial p}{\partial x}$$ (3) **Note that this is valid for variable area too**

Decrease in pressure results in increase of velocity

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I-Law of Thermodynamics

- At steady state

$$0 = \dot{Q}_{CV} - \dot{W}_{CV} + \dot{m}\left((h_i - h_e) + \frac{V_i^2 - V_e^2}{2} + g(z_i - z_e)\right)$$

$$\Rightarrow h_i + \frac{V_i^2}{2} = h_e + \frac{V_e^2}{2} \qquad \Rightarrow dh + d\left(\frac{V^2}{2}\right) = 0$$

$$\Rightarrow c_p dT + d\left(\frac{V^2}{2}\right) = 0$$ (4)

Increase in velocity results in decrease of Temperature

Equation of State

Equation of State

$$p = \rho RT \qquad \Rightarrow \frac{p}{\rho T} = R$$

$$\Rightarrow \frac{dp}{\rho T} - \frac{p}{T}\frac{d\rho}{\rho^2} - \frac{p}{\rho}\frac{dT}{T^2} = 0$$

$$\Rightarrow \frac{dp}{p} - \frac{d\rho}{\rho} - \frac{dT}{T} = 0 \qquad (5)$$

Mach Number

- In compressible flow it is useful to unify all the results in terms of Mach Number
- Mach Number is defined as the ratio of fluid velocity and the local sonic speed $\Rightarrow M = V/c$ (6)
- Thus the Mach Cone angle can be written as $\alpha = \text{Sin}^{-1}(1/M)$
- All property changes can be related to local Mach number as follows

Momentum balance [Eq. (3)]

$$\Rightarrow V\frac{dV}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \Rightarrow \rho V dV = -dp \qquad \Rightarrow \frac{dp}{p} = -\frac{\rho V^2}{p}\frac{dV}{V}$$

Since $\frac{\rho}{p} = \frac{\gamma}{c^2}$

$$\Rightarrow \frac{dp}{p} = -\gamma\frac{V^2}{c^2}\frac{dV}{V} = -\gamma M^2\frac{dV}{V} \qquad (7)$$

Property Relations-I

Energy balance [Eq. (4)] $\Rightarrow c_p dT + d(V^2/2) = 0$

$$\Rightarrow dT = -V\frac{dV}{c_p} \qquad \Rightarrow \frac{dT}{T} = -\frac{1}{T}\frac{V^2}{c_p}\frac{dV}{V} \qquad \Rightarrow \frac{dT}{T} = -\frac{\gamma R}{c^2}\frac{V^2}{c_p}\frac{dV}{V}$$

Note $\Rightarrow \frac{\gamma R}{c_p} = \frac{\gamma(c_p - c_v)}{c_p} = \gamma\left(\frac{1}{\gamma} - 1\right) = \gamma - 1$

$$\therefore \frac{dT}{T} = -(\gamma - 1)M^2\frac{dV}{V} \qquad (8)$$

Equation of State [Eq. (5)]

$$\Rightarrow \frac{d\rho}{\rho} = \frac{dp}{p} - \frac{dT}{T} \qquad \Rightarrow \frac{d\rho}{\rho} = \frac{dV}{V}\left(-\gamma M^2 + (\gamma - 1)M^2\right)$$

$$\Rightarrow \frac{d\rho}{\rho} = -M^2\frac{dV}{V} \qquad (9)$$

Property Relations-II

Finally Mass balance [Eq. (2)]

$$\Rightarrow \frac{dV}{V} + \frac{d\rho}{\rho} = -\frac{dA}{A}$$

Equation (15) $\Rightarrow \frac{d\rho}{\rho} = -M^2\frac{dV}{V}$

$$(1 - M^2)\frac{dV}{V} = -\frac{dA}{A} \qquad (10)$$

- We shall derive many general conclusions from all the relations derived above
- Equation (9) implies that for M< 0.3, fractional change in density is less than 9% of fractional change in velocity.

Property Relations-III

- This is the general incompressibility condition commonly used
- Equation (8) implies that for M< 0.3, fractional change in Temperature for air is less than 4% of fractional change in velocity
- Most importantly, equation (10) underlines the characteristic difference between subsonic (M<1) and Supersonic flows (M>1)
- This is discussed in next slide

Property Relations-IV

Eq. (10) $\Rightarrow \frac{dV}{V} = -\frac{dA}{A}\frac{1}{(1-M^2)}$

If M < 1 or 1-M² > 0, then dV > 0 for dA < 0 and dV < 0 for dA > 0

- The above conditions imply that the nozzle will be converging and the diffuser will be diverging under subsonic conditions
- The opposite, i.e., nozzle will be diverging and diffuser will be converging under supersonic conditions is given by the condition stated below

If M > 1 or 1-M² < 0, then dV < 0 for dA < 0 and dV > 0 for dA > 0

Stagnation Properties

- When a fluid is brought to rest from a given state under isentropic conditions, the resulting properties are called stagnation properties
- These are denoted with the subscript '0'
- Thus, h_0, p_0, T_0, ρ_0 are stagnation enthalpy, pressure, temperature and density
- The following results can be written:

$\Rightarrow h_0 = h + \frac{V^2}{2}$ **Comes from energy balance** (11)

$\Rightarrow T_0 = T + \frac{V^2}{2c_p}$ **Ideal gas and constant specific heat** (12)

- We shall now show the relations between two points in isentropic flow

Isentropic Flow-I

- For isentropic flow, We can write

$$p_1 v_1^{\gamma} = p_2 v_2^{\gamma} \qquad \left(\frac{T_2}{T_1}\right) = \left(\frac{v_1}{v_2}\right)^{\gamma-1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} \qquad \left(\frac{T_2}{T_1}\right) = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

- Since $c = \sqrt{\gamma RT} \Rightarrow \frac{c_2}{c_1} = \left(\frac{T_2}{T_1}\right)^{0.5} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{2\gamma}} = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{\gamma-1}{2}}$
- Energy equation $\Rightarrow h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$

$$\Rightarrow c_p T_1 + \frac{V_1^2}{2} = c_p T_2 + \frac{V_2^2}{2} \quad \Rightarrow T_1 + \frac{V_1^2}{2c_p} = T_2 + \frac{V_2^2}{2c_p} = T_0$$

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Isentropic Flow-II

$$\Rightarrow \frac{T_1 + \frac{V_1^2}{2c_p}}{T_2 + \frac{V_2^2}{2c_p}} = 1 \qquad \Rightarrow \frac{T_1\left(1 + \frac{V_1^2}{2c_pT_1}\right)}{T_2\left(1 + \frac{V_2^2}{2c_pT_2}\right)} = 1$$

$$\frac{V^2}{2c_pT} = \frac{V^2}{2c_p\frac{c^2}{\gamma R}} = \frac{M^2\gamma R}{2c_p} = \frac{M^2c_p/c_v(c_p - c_v)}{2c_p} = \frac{M^2(\gamma-1)}{2}$$

$$\Rightarrow \frac{1+\left(\frac{\gamma-1}{2}\right)M_1^2}{1+\left(\frac{\gamma-1}{2}\right)M_2^2} = \frac{T_2}{T_1} \qquad (13)$$

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Isentropic Flow-III

- Since $\left(\frac{\rho_2}{\rho_1}\right) = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}}$

$$\Rightarrow \frac{\rho_2}{\rho_1} = \left[\frac{1+\left(\frac{\gamma-1}{2}\right)M_1^2}{1+\left(\frac{\gamma-1}{2}\right)M_2^2}\right]^{\frac{1}{\gamma-1}} \qquad (14)$$

- Similarly, since $\left(\frac{p_2}{p_1}\right) = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$

$$\Rightarrow \frac{p_2}{p_1} = \left[\frac{1+\left(\frac{\gamma-1}{2}\right)M_1^2}{1+\left(\frac{\gamma-1}{2}\right)M_2^2}\right]^{\frac{\gamma}{\gamma-1}} \qquad (15)$$

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Isentropic Flow-IV

- If point 2 is the stagnation point and point 1 is any general point, then

$$\Rightarrow \frac{T_0}{T} = 1+\left(\frac{\gamma-1}{2}\right)M^2 \qquad \Rightarrow \frac{c_0}{c} = \left[1+\left(\frac{\gamma-1}{2}\right)M^2\right]^{0.5} \qquad (16)$$

$$\Rightarrow \frac{\rho_0}{\rho} = \left[1+\left(\frac{\gamma-1}{2}\right)M^2\right]^{\frac{1}{\gamma-1}} \qquad (17)$$

$$\Rightarrow \frac{p_0}{p} = \left[1+\left(\frac{\gamma-1}{2}\right)M^2\right]^{\frac{\gamma}{\gamma-1}} \qquad (18)$$

- For air (γ=1.4), the values of T_0/T, c_0/c, ρ_0/ρ, p_0/p are tabulated as a function of Mach number

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Concept of Static and Stagnation Pressure-I

- **Static Pressure:** Local thermodynamic pressure measured without changing its state
- **Stagnation Pressure:** It is the hypothetical pressure that will be measured, if the fluid is brought to rest in a frictionless manner at the same elevation

The relationship between static and stagnation pressure for incompressible flow can be obtained by Bernoulli's equation

$$p_1 + \frac{\rho V_1^2}{2} + \rho g H_1 = p_2 + \frac{\rho V_2^2}{2} + \rho g H_2$$

$\frac{\rho V_2^2}{2} \to 0$; $H_1 = H_2$

$$\therefore p_{stagnation} = p_{static} + \frac{\rho V^2}{2}$$

The term $\frac{\rho V^2}{2}$ is called dynamic pressure

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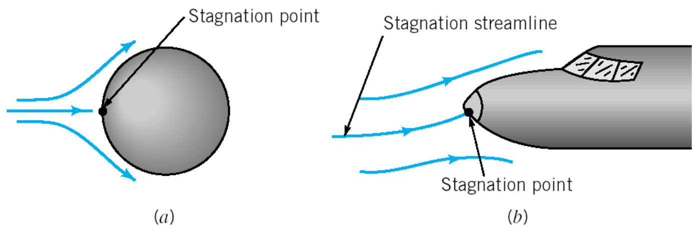
Concept of Static and Stagnation Pressure-II



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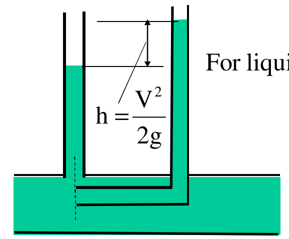
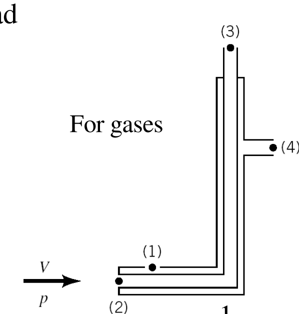
Pitot Tube-I

- Local fluid velocity can be estimated, if static and stagnation pressures can be measured.
- Concept of piezometric head



For liquids

For gases



$$p_3 = p_1 + \frac{1}{2}\rho V^2$$

This can be done by using Pitot-static tube

$$p_4 = p_1$$

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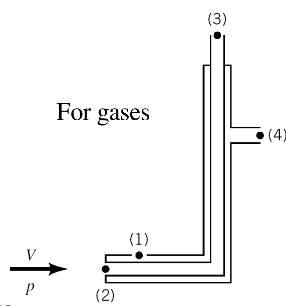
Pitot Tube-II

- The same concepts can be extended to compressible flow
- p_3 is the stagnation pressure
- p_4 is the local static pressure
- In slide 18, we have shown that

$$\Rightarrow \frac{p_0}{p} = \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]^{\frac{\gamma}{\gamma - 1}}$$

- Thus by knowing p and p_o, we can measure the Mach number

For gases



$$p_3 = p_0$$

$$p_4 = p$$

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Isentropic Flow-V

- If the Mach number at the general point is 1, then the condition of the fluid is called critical.
- The properties at critical state are denoted with a * in the superscript
- We can derive the following relations between the critical and stagnation properties

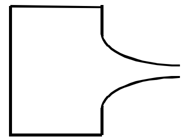
$$\Rightarrow \frac{T_0}{T^*} = 1 + \left(\frac{\gamma - 1}{2}\right) = \frac{\gamma + 1}{2} \quad (19)$$

$$\Rightarrow \frac{\rho_0}{\rho^*} = \left[\frac{\gamma + 1}{2}\right]^{\frac{1}{\gamma - 1}} \quad (20)$$

$$\Rightarrow \frac{p_0}{p^*} = \left[\frac{\gamma + 1}{2}\right]^{\frac{\gamma}{\gamma - 1}} \quad (21)$$

Flow in a Variable Area Passage-I

- We had shown earlier that

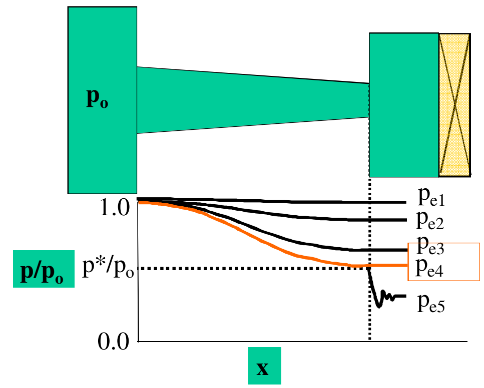


Eq. (10) $\Rightarrow \frac{dV}{V} = -\frac{dA}{A}\frac{1}{(1-M^2)}$

- This implies that if we have a converging nozzle and initial Mach number is less than 1, then it will accelerate till Mach number becomes one.
- It should be understood that for a given p_0 and T_0, the boundary condition at the other end shall either be p_{amb} or M = 1.
- Let us understand the flow characteristics qualitatively

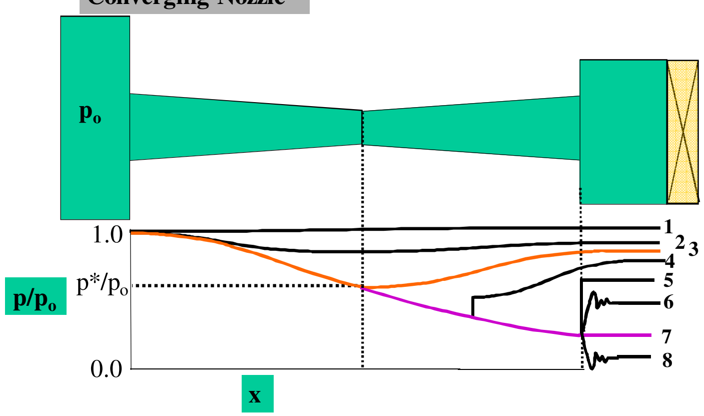
Flow in a Variable Area Passage-II

Converging Nozzle



Flow in a Variable Area Passage-III

Converging Nozzle



Flow in a Variable Area Passage-IV

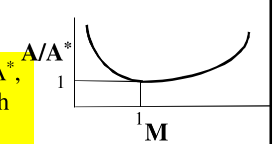
- To get the ratio between A/A^*, we invoke mass balance

$$\rho AV = \rho^* A^* V^* \Rightarrow \frac{A}{A^*} = \frac{\rho^* V^*}{\rho V} = \frac{\rho^* c^*}{\rho M c} = \frac{1}{M}\frac{\rho^*/\rho_0}{\rho/\rho_0}\sqrt{\frac{T^*/T_0}{T/T_0}}$$

$$\Rightarrow \frac{A}{A^*} = \frac{1}{M}\frac{\rho_0/\rho}{\rho_0/\rho^*}\sqrt{\frac{T_0/T}{T_0/T^*}} = \frac{1}{M}\left[\frac{1+\left(\frac{\gamma-1}{2}\right)M^2}{1+\left(\frac{\gamma-1}{2}\right)}\right]^{\frac{1}{\gamma-1}}\left[\frac{1+\left(\frac{\gamma-1}{2}\right)M^2}{1+\left(\frac{\gamma-1}{2}\right)}\right]^{\frac{1}{2}}$$

$$\Rightarrow \frac{A}{A^*} = \frac{1}{M}\left[\frac{1+\left(\frac{\gamma-1}{2}\right)M^2}{1+\left(\frac{\gamma-1}{2}\right)}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad (28)$$

For a given A/A^*, we have 2 Mach Numbers



Flow in a Variable Area Passage-V

- Several general observations can be made
- In isentropic flow, A/A^* is also a function of Mach number only
- A^* implies area where Mach number is 1
- In a converging nozzle it is the smallest area
- In a converging-diverging (CD) nozzle, it will be the throat
- A nozzle is said to be choked when M = 1 at the throat.
- Mass flow rate reaches a maximum when a nozzle is choked (for a given stagnation condition)

Flow in a Variable Area Passage-VII

- The mass flow can be estimated as follows

$$\Rightarrow h_0 = h + \frac{V^2}{2} \qquad \Rightarrow V^2 = 2c_p(T_0 - T)$$

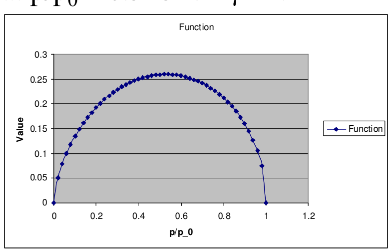
$$\Rightarrow V^2 = 2c_pT_0\left(1-\frac{T}{T_0}\right) \qquad \Rightarrow V^2 = 2c_pT_0\left(1-\left[\frac{p}{p_0}\right]^{\frac{\gamma-1}{\gamma}}\right)$$

$$\dot{m} = A\rho V = A\rho_0\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}\left(2c_pT_0\left(1-\left[\frac{p}{p_0}\right]^{\frac{\gamma-1}{\gamma}}\right)\right)^{0.5}$$

$$\dot{m} = A\rho_0\left(2c_pT_0\left(\left[\frac{p}{p_0}\right]^{\frac{2}{\gamma}}-\left[\frac{p}{p_0}\right]^{\frac{\gamma+1}{\gamma}}\right)\right)^{0.5} \qquad (29)$$

Flow in a Variable Area Passage-VIII

- For given p_0 and ρ_0, mass flow rate given by Eq. (29) in the previous slide is a function of p/p_0
- The function in the parenthesis, when plotted gives maxima at $p/p_0 = 0.528$ for $\gamma = 1.4$



- The same can be analytically obtained by differentiating Eq. (29) and equating to 0, leading to

Flow in a Variable Area Passage-IX

$$\frac{p}{p_0} = \left[\frac{2}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}} = 0.528 \qquad \textbf{For } \gamma = \textbf{1.4}$$

- The expression for maximum mass flow can now be obtained by substituting for p/p_0 from above in Eq. (29) expressing c_p in terms of γ and R

$$\Rightarrow \dot{m}_{max} = A_t\rho_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\sqrt{\gamma RT_0} \qquad (30)$$

- Thus, if the stagnation conditions are known, then the nozzle can be designed for a desired mass flow rate

- We can recast the mass flow rate equation (Eq.(29) in the following form. Note the reference is throat.

$$\dot m = A_t\rho_0\left(2\frac{\gamma R}{\gamma-1}T_0\left(\left[\frac{p_t}{p_0}\right]^{\frac{2}{\gamma}}-\left[\frac{p_t}{p_0}\right]^{\frac{\gamma+1}{\gamma}}\right)\right)^{0.5}$$

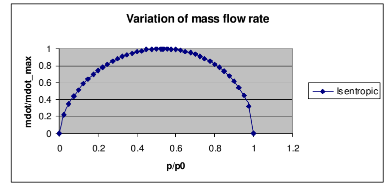
$$\Rightarrow \dot m_{max} = A_t\rho_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\sqrt{\gamma R T_0}$$

From Eq. (30)

$$\Rightarrow \frac{\dot m}{\dot m_{max}} = \frac{\left(\frac{2}{\gamma-1}\left(\left[\frac{p_t}{p_0}\right]^{\frac{2}{\gamma}}-\left[\frac{p_t}{p_0}\right]^{\frac{\gamma+1}{\gamma}}\right)\right)^{0.5}}{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}$$

From the above two equations

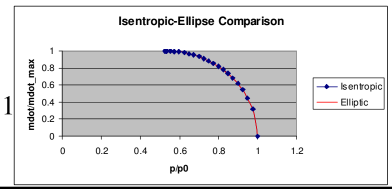
- If we plot the non-dimensional mass flow rate equation we get the following variation



- The right side of the curve for 0.528 > p/po > 1, the curve can very closely represented by an ellipse given by

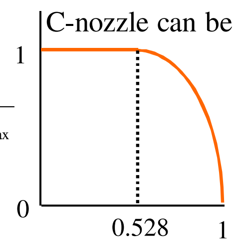
$$\Rightarrow \frac{\left[\frac{p_t}{p_0}-0.528\right]^2}{0.472^2}+\frac{\left(\frac{\dot m}{\dot m_{max}}\right)^2}{1}=1$$

Error less than 0.5%

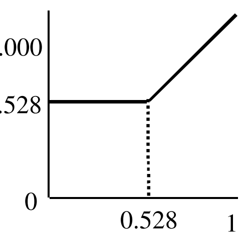


Flow in a Variable Area Passage-X

- Before concluding, let us visit C- nozzle again
- The non-dimensional mass flow rate for C-nozzle can be drawn
- The variation of pressure at exit plane and at back of C-nozzle can be as shown



$\frac{\dot m}{\dot m_{max}}$, $\frac{p_b}{p_0}$



$\frac{p_e}{p_0}$, $\frac{p_b}{p_0}$

Normal Shock-I

- We saw that a normal shock can arise in a CD nozzle
- During normal shock there are jump in properties
- We shall establish relation ships in properties before and after a normal shock

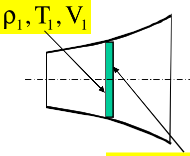
Mass balance

$$\Rightarrow \rho_1 V_1 = \rho_2 V_2$$

Energy balance

$$\Rightarrow h_1 + 0.5V_1^2 = h_2 + 0.5V_2^2$$

$$\Rightarrow h_{01} = h_{02} \Rightarrow T_{01} = T_{02}$$



p_1, ρ_1, T_1, V_1

p_2, ρ_2, T_2, V_2

Normal Shock-II

- Since flow is isentropic on both sides of the shock (except at the shock front)

$$\Rightarrow \frac{T_{01}}{T_1} = 1 + \left(\frac{\gamma-1}{2}\right) M_1^{\,2}; \quad \frac{T_{02}}{T_2} = 1 + \left(\frac{\gamma-1}{2}\right) M_2^{\,2}$$

$$\Rightarrow \frac{T_2}{T_1} = \frac{1 + \left(\frac{\gamma-1}{2}\right) M_1^{\,2}}{1 + \left(\frac{\gamma-1}{2}\right) M_2^{\,2}} \quad (31)$$

As $T_{01} = T_{02}$ (from previous slide)

- From Continuity,

$$\Rightarrow \rho_1 V_1 = \rho_2 V_2 \Rightarrow \frac{p_1}{RT_1} V_1 = \frac{p_2}{RT_2} V_2$$

Normal Shock-III

$$\Rightarrow \frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{V_2}{V_1} = \frac{p_2}{p_1}\frac{M_2 c_2}{M_1 c_1} = \frac{p_2}{p_1}\frac{M_2\sqrt{T_2}}{M_1\sqrt{T_1}}$$

$$\therefore \frac{\sqrt{T_2}}{\sqrt{T_1}} = \frac{p_2}{p_1}\frac{M_2}{M_1} \Rightarrow \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\frac{M_2}{M_1}\right)^2 \quad (32)$$

Eqs (31) and (32)
$$\Rightarrow \frac{1 + \left(\frac{\gamma-1}{2}\right) M_1^{\,2}}{1 + \left(\frac{\gamma-1}{2}\right) M_2^{\,2}} = \left(\frac{p_2}{p_1}\frac{M_2}{M_1}\right)^2$$

$$\Rightarrow \frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{1 + \left(\frac{\gamma-1}{2}\right) M_1^{\,2}}{1 + \left(\frac{\gamma-1}{2}\right) M_2^{\,2}}\right)^{0.5} \quad (33)$$

Normal Shock-IV

Momentum balance

$$\Rightarrow p_1 - p_2 = \rho_2 V_2^{\,2} - \rho_1 V_1^{\,2} \quad \Rightarrow p_1 + \rho_1 V_1^{\,2} = p_2 + \rho_2 V_2^{\,2}$$

$$\Rightarrow p_1 + \rho_1 M_1^{\,2} c_1^{\,2} = p_2 + \rho_2 M_2^{\,2} c_2^{\,2}$$

$$\Rightarrow p_1 + \frac{p_1}{\cancel{RT_1}} M_1^{\,2}\gamma \cancel{RT_1} = p_2 + \frac{p_2}{\cancel{RT_2}} M_2^{\,2}\gamma \cancel{RT_2}$$

$$\Rightarrow \frac{p_2}{p_1} = \frac{1+\gamma M_1^{\,2}}{1+\gamma M_2^{\,2}} \quad (34)$$

Eqs (33) and (34)
$$\Rightarrow \frac{M_1}{M_2}\left(\frac{1 + \left(\frac{\gamma-1}{2}\right) M_1^{\,2}}{1 + \left(\frac{\gamma-1}{2}\right) M_2^{\,2}}\right)^{0.5} = \frac{1+\gamma M_1^{\,2}}{1+\gamma M_2^{\,2}}$$

Normal Shock-V

- Squaring both sides, cross multiplying and simplifying leads to

$$M_2 = \left(\frac{M_1^{\,2} + \left(\frac{2}{\gamma-1}\right)}{\left(\frac{2\gamma}{\gamma-1}\right) M_1^{\,2} - 1}\right)^{0.5} \quad (35)$$

- It is now straightforward to derive the following

$$\Rightarrow \frac{T_2}{T_1} = \frac{1 + \frac{\gamma-1}{2} M_1^{\,2}}{1 + \frac{\gamma-1}{2} M_2^{\,2}} \quad (36)$$

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Normal Shock-VI

$$\Rightarrow \frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left(\frac{1+\frac{\gamma-1}{2}M_2^{\,2}}{1+\frac{\gamma-1}{2}M_1^{\,2}}\right)^{0.5}$$ (37)

$$\Rightarrow \frac{p_{02}}{p_{01}} = \left(\frac{1+\frac{\gamma-1}{2}M_2^{\,2}}{1+\frac{\gamma-1}{2}M_1^{\,2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\frac{M_1}{M_2}$$ (38)

$$\Rightarrow \frac{A^*_1}{A^*_2} = \frac{p_{02}}{p_{01}}$$ (39)

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