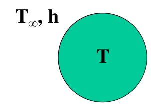
Transient Heat Transfer

- Having gotten a feel for the steady state heat transfer, let us get a feel for the transient behavior of heat transfer
- To begin with, let us assume that spatial variation of temperature is negligible and the temperature of the body as a whole changes with time
- This implies that the thermal conductivity is very high.
- This type of problem is called lumped analysis
- We shall evolve a criterion for the validity of this assumption as we go along
- Let us look at the steps in analysis

Lumped Analysis-I

- Let the body temperature be denoted by T
- The body interacts with the surroundings with the temperature at T_∞ and heat transfer coefficient h



- First law implies that $\dot{E} = \dot{Q} - \dot{W}$

$$\Rightarrow \frac{d}{dt}(mcT) = -hA(T - T_\infty)$$

- For constant specific heat

$$\Rightarrow mc\frac{dT}{dt} = -hA(T - T_\infty)$$

Lumped Analysis-II

- Defining $\theta = T - T_\infty$, we can write

$$\Rightarrow mc\frac{d\theta}{dt} = -hA\theta \quad \Rightarrow \frac{d\theta}{dt} = -\frac{hA}{mc}\theta$$

Let $\theta = \theta_o$ at $t = 0$

- The solution for the above equation is

$$\theta = \theta_0 e^{-\frac{hA}{mc}t}$$

- If the object is sphere, we have

$$m = \frac{4}{3}\pi R^3 \rho; \quad A = 4\pi R^2 \quad \Rightarrow \frac{hA}{mc} = \frac{h}{c}\frac{4\pi R^2}{\frac{4}{3}\pi R^3 \rho} = \frac{h}{\rho c}\frac{3}{R}$$

$$\theta = \theta_0 e^{-\frac{h}{\rho c}\frac{3}{R}t}$$

Lumped Analysis-III

- In general $\theta = \theta_0 e^{-\frac{hA}{mc}t} = \theta_0 e^{-\frac{hA}{V\rho c}t} = \theta_0 e^{-\frac{h}{\rho c}\frac{A}{V}t}$
- If the object is a cylinder of Radius R and Length L, we have

$$\Rightarrow \frac{A}{V} = \frac{2\pi RL + 2\pi R^2}{\pi R^2 L} = \frac{2(L+R)}{RL}$$

$$\theta = \theta_0 e^{-\frac{h}{\rho c}\frac{2(L+R)}{LR}t}$$

- If the cylinder has L>>R , then

$$\theta = \theta_0 e^{-\frac{h}{\rho c}\frac{2}{R}t}$$

Lumped Analysis-III

- This concept can now be extended to any geometry
- To generalize the result, it is customary to introduce the characteristic length L
- The logic suggests that V/A is the most obvious choice

$$\Rightarrow \theta = \theta_0 e^{-\frac{h}{\rho c L}t} = \theta_0 e^{-\frac{hL}{k}\frac{k}{\rho c L^2}t} = \theta_0 e^{-\frac{hL}{k}\frac{\alpha t}{L^2}}$$

- In the above expression, we have introduced the property called thermal diffusivity = $k/(\rho c)$
- The non-dimensional parameter hL/k and $\alpha t/L^2$ are called the Biot Number and Fourier No respectively

Lumped Analysis-IV

- Thus, the temperature variation is a function of two non dimensional parameters Biot number and Fourier number.
- We will appreciate these parameters, as we go into more complex cases
- We can give a physical interpretation for the Biot number as follows

$$Bi = \frac{hL}{K} = \frac{L}{KA}\frac{hA}{1} = \frac{\text{conduction Resistance}}{\text{convection Resistance}}$$

- When Bi is very small, it implies that conduction resistance is very small and hence lumped analysis valid
- The criterion used is $Bi < 0.1$

Transients with spatial effects

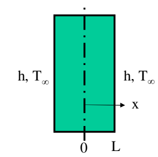
- If Bi > 0.1 spatial effects become important and so more complications are involved
- Exact analytical solutions can be obtained using separation of variables similar to 2-D steady state analysis
- Let us look at 1-D transient analysis in a slab geometry with no heat generation
- The governing equation for this case is

$$\rho c\frac{\partial (T)}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \cancel{\frac{\partial^2 T}{\partial y^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}}\right) + \cancel{q'''} \quad \Rightarrow \frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

1-D Transient in a Plate-I

- The governing equation

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad 0 \le x \le L;\ 0 \le t$$



- Boundary conditions

$T(0,x) = T_i$; $[\partial T/\partial x](t,0) = 0$; $[-k\partial T/\partial x](t,L) = h(T(t,L) - T_\infty)$

- Let us define $\theta = T - T_\infty$

$$\Rightarrow \frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} \qquad 0 \le x \le L;\ 0 \le t$$

$[\partial\theta/\partial x](t,0) = 0$; $-k[\partial\theta/\partial x](t,L) = h\theta(t,L)$;
$\theta(0,x) = \theta_0$

1-D Transient in a Plate-II

- The solution for $\theta(t,x)$ is assumed of the form:

$$\theta(t,x) = X(x)T(t)$$

- Substituting this in the governing equation, we get

$$\Rightarrow X(x)\frac{1}{\alpha}\frac{dT(t)}{dt} = T(t)\frac{d^2X(x)}{dx^2} = 0 \qquad \Rightarrow \frac{1}{\alpha}\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

- Since LHS is only a function of t and RHS is only a function of x and yet they be equal, would require that both sides be equal to a constant

$$\Rightarrow \frac{1}{\alpha}\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \quad \Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0; \quad \frac{dT}{dt} - \alpha\lambda^2 T = 0$$

1-D Transient in a Plate-II

- The solution for X and T are;

$$X = C_1\cos(\lambda x) + C_2\sin(\lambda x) \qquad T = C_3 e^{-\alpha\lambda^2 t}$$

$$\Rightarrow \theta = XT = (C_1\cos(\lambda x) + C_2\sin(\lambda x))\left(C_3 e^{-\alpha\lambda^2 t}\right)$$
$$= (C_1\cos(\lambda x) + C_2\sin(\lambda x))\left(e^{-\alpha\lambda^2 t}\right)$$

C_3 is absorbed in C_1 and C_2

- The BC at x = 0; $\partial\theta/\partial x = 0$, requires a symmetric solution and hence $C_2 = 0$

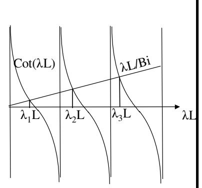
$$\Rightarrow \theta = (C_1\cos(\lambda x))\left(e^{-\alpha\lambda^2 t}\right) \Rightarrow \frac{\partial\theta}{\partial x} = (C_1\lambda(-\sin(\lambda x))\left(e^{-\alpha\lambda^2 t}\right)$$

BC at x = L $\Rightarrow \left(-k\cancel{C_1}(-\lambda)\sin(\lambda L)\right)\cancel{\left(e^{-\alpha\lambda^2 t}\right)} = h\left(\cancel{C_1}\cos(\lambda L)\right)\cancel{\left(e^{-\alpha\lambda^2 t}\right)}$

$$\Rightarrow (\lambda\sin(\lambda L)) = \frac{h}{k}(\cos(\lambda L)) \qquad \Rightarrow \cot(\lambda L) = \frac{k\lambda}{h} = \frac{\lambda L}{Bi}$$

1-D Transient in a Plate-III

- The solution for the eigen values can be graphically interpreted as shown
- Thus we will have infinite values of eigen values, but these have to be numerically determined. Your book summarizes the first four values as a function of Bi in Appendix B3



- Thus the solution for θ can be written as

$$\Rightarrow \theta = \sum_{n=1}^{\infty} C_n e^{-\alpha\lambda_n^2 t}\cos(\lambda_n x)$$

Still C_n needs to be determined

1-D Transient in a Plate-IV

- The constants have to be determined from initial condition
- This will be again done by orthogonal functions
- It turns out that

$$\int_0^L \cos(\lambda_n x)\cos(\lambda_m x)dx = 0 \qquad \text{for } n \ne m$$

Messy to prove

Will be given as home work

$$\int_0^L \cos^2(\lambda_n x)dx = \frac{\lambda_n L + \sin(\lambda_n L)\cos(\lambda_n L)}{2\lambda_n}$$

$$\int_0^L \cos(\lambda_n x)dx = \frac{\sin(\lambda_n L)}{\lambda_n}$$

1-D Transient in a Plate-V

- The initial condition is $\theta(0,x) = \theta_0$

$$\Rightarrow \theta_0 = \sum_{n=1}^{\infty} C_n \cos(\lambda_n x)$$

$$\int_0^L \theta_0 \cos(\lambda_m x)dx = \int_0^L \sum_{n=1}^{\infty} C_n \cos(\lambda_m x)\cos(\lambda_n x)dx$$

- Note that in the RHS, only term n=m will survive

$$\int_0^L \theta_0 \cos(\lambda_n x)dx = C_n \int_0^L \cos^2(\lambda_n x)dx$$

$$\theta_0 \frac{\sin(\lambda_n L)}{\lambda_n} = C_n \frac{\lambda_n L + \sin(\lambda_n L)\cos(\lambda_n L)}{2\lambda_n}$$

$$\therefore C_n = \theta_0 \frac{2\sin(\lambda_n L)}{\lambda_n L + \sin(\lambda_n L)\cos(\lambda_n L)}$$

1-D Transient in a Plate-VI

- Thus the solution for θ can be written as

$$\Rightarrow \frac{\theta}{\theta_0} = 2\sum_{n=1}^{\infty} \frac{\sin(\lambda_n L)}{\lambda_n L + \sin(\lambda_n L)\cos(\lambda_n L)} e^{-\alpha\lambda_n^2 t}\cos(\lambda_n x)$$

$$\Rightarrow \frac{\theta}{\theta_0} = 2\sum_{n=1}^{\infty} \frac{\sin(\lambda_n L)}{\lambda_n L + \sin(\lambda_n L)\cos(\lambda_n L)} e^{-\alpha\lambda_n^2 t}\cos(\lambda_n L \frac{x}{L})$$

$$\Rightarrow \frac{\theta}{\theta_0} = f\left(\lambda_n L, \frac{\alpha t}{L^2}, \frac{x}{L}\right)$$

- But, $\lambda_n L$ is determined by Bi

$$\therefore \frac{\theta}{\theta_0} = f\left(Bi, \frac{\alpha t}{L^2}, \frac{x}{L}\right) = f\left(Bi, Fo, \frac{x}{L}\right)$$

Heisler Charts-I

- The solution for θ was shown to be

$$\Rightarrow \frac{\theta}{\theta_0} = 2\sum_{n=1}^{\infty} \frac{\sin(\lambda_n L)}{\lambda_n L + \sin(\lambda_n L)\cos(\lambda_n L)} e^{-\alpha\lambda_n^2 t}\cos(\lambda_n x)$$

- In functional form

$$\frac{\theta}{\theta_0} = f\left(Bi, \frac{\alpha t}{L^2}, \frac{x}{L}\right) = f\left(Bi, Fo, \frac{x}{L}\right)$$

- Heisler showed that when Fo > 0.2, just one term is adequate to describe the solution

$$\Rightarrow \frac{\theta}{\theta_0} = C_1 e^{-\alpha\lambda_1^2 t}\cos(\lambda_1 x) \quad \text{Where } C_1 = 2\frac{\sin(\lambda_1 L)}{\lambda_1 L + \sin(\lambda_1 L)\cos(\lambda_1 L)}$$

Heisler Charts-II

- The coordinate of the mid-plane is x =0

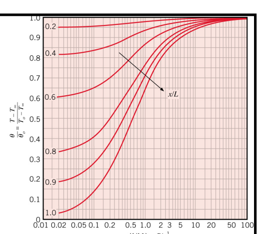
$$\Rightarrow \frac{\theta(0)}{\theta_0} = C_1 e^{-\alpha\lambda_1^2 t} = C_1 e^{-(\lambda_1 L)^2 Fo} = f(Fo, \lambda_1 L) = f(Fo, Bi)$$

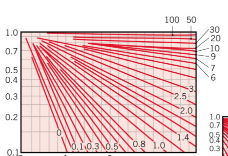
- Thus, the spatial profile at any time can be written as

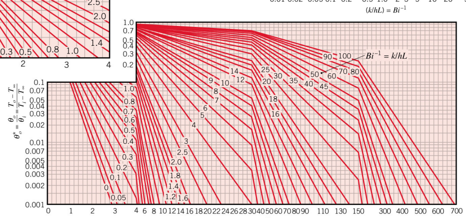
$$\Rightarrow \frac{\theta}{\theta(0)} = \cos(\lambda_1 x) = f(\lambda_1 L, x/L) = f(Bi, x/L)$$

- Heisler presented these results in the form of two graphs, called Heisler Charts
- For Cartesian system, it is shown in the next slide. Similar curves for the cylindrical and spherical one dimensional cases are given in Appendix D of your book

Heisler Charts-III







Criterion for Lumped Analysis

- If the variation of temperature within the slab is less than 5%, we can call it lumped

$$\Rightarrow \frac{\theta(x=L)}{\theta(0)} = \cos(\lambda_1 L) = 0.95 \qquad \Rightarrow \lambda_1 L = 0.318 \text{ radian}$$

$$\Rightarrow Bi = \frac{\lambda_1 L}{\cot(\lambda_1 L)} = 0.1$$

- Thus, Bi should be < 0.1 for lumped analysis to be valid. This can also be viewed in the chart

Energy Storage-I

$\theta = \theta_0 C_1 e^{-\alpha\lambda_1^2 t}\cos(\lambda_1 x)$

- As energy storage is one of the main application of transient, it is useful to get a method to estimate the total energy stored
- From thermodynamics

$$\Delta E = 2\int_0^L \rho Ac(T-T_0)dx = 2\rho AcL\frac{1}{L}\int_0^L(\theta-\theta_0)dx$$

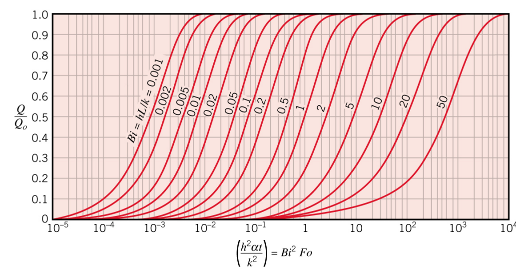
$$= 2\rho AcL\theta_0\frac{1}{L}\int_0^L(C_1 e^{-\alpha\lambda_1^2 t}\cos(\lambda_1 x)-1)dx$$

$$\frac{\Delta E}{2\rho AcL\theta_0} = \frac{1}{L}\left(C_1 e^{-\alpha\lambda_1^2 t}\frac{\sin(\lambda_1 x)}{\lambda_1}-x\right)_0^L$$

$$\frac{\Delta E}{\Delta E_0} = \left(C_1 e^{-\alpha\lambda_1^2 t}\frac{\sin(\lambda_1 L)}{\lambda_1 L}-1\right) = \left(\theta(0)\frac{\sin(\lambda_1 L)}{\lambda_1 L}-1\right)$$

Energy Storage-II

$$\frac{\Delta E}{\Delta E_0} = \left(\theta(0)\frac{\sin(\lambda_1 L)}{\lambda_1 L}-1\right) = f(\lambda_1 L, Fo) = f(Bi, Fo)$$



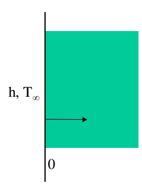
1-D Transient in a Semi-Infinite Plate-I

- Another case of interest in transient heat transfer is what is called heat transfer in a Semi-infinite plate
- It is has many useful applications. In fact, all transient problems start as semi-infinite wall
- The governing equation

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad 0 \le x \le \infty;\ 0 \le t$$

- Boundary conditions

$T(0,x) = T_i$; $T(t,0) = T_s$; $T(t,x\to\infty) = T_i$



1-D Transient in a Semi-Infinite Plate-II

- Mathematically, problems that have boundary at infinity are often solved by a method called similarity solution
- In this method, a new variable, called similarity variable is introduced
- This variable is chosen such that T becomes a function of only this variable
- Thus the governing equation will be transformed into an ODE from PDE
- There are systematic ways by which this can be derived, but often involves some qualitative arguments

1-D Transient in a Semi-Infinite Plate-III

- In this course we shall give you the form of the variable. Note that it will be a combination of x and t

$$\eta = \frac{x}{(4\alpha t)^{0.5}} \quad \Rightarrow \frac{\partial\eta}{\partial x} = \frac{1}{(4\alpha t)^{0.5}}\ ,\ \frac{\partial\eta}{\partial t} = \frac{x}{(4\alpha)^{0.5}}\frac{-0.5}{t^{1.5}}$$

- Using chain rule

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial\eta}{\partial x} = \frac{dT}{d\eta}\frac{1}{(4\alpha t)^{0.5}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{dT}{d\eta}\frac{1}{(4\alpha t)^{0.5}}\right) = \frac{d^2T}{d\eta^2}\frac{\partial\eta}{\partial x}\frac{1}{(4\alpha t)^{0.5}} = \frac{d^2T}{d\eta^2}\frac{1}{(4\alpha t)}$$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial\eta}{\partial t} = \frac{dT}{d\eta}\frac{x}{(4\alpha)^{0.5}}\frac{-0.5}{(t)^{1.5}} = \frac{dT}{d\eta}\frac{-x}{2t(4\alpha t)^{0.5}}$$

1-D Transient in a Semi-Infinite Plate-IV

- Substituting for the partial derivatives in the heat equation, we get

$$\frac{1}{\alpha}\frac{dT}{d\eta}\frac{-x}{2t(4\alpha t)^{0.5}} = \frac{d^2T}{d\eta^2}\frac{1}{(4\alpha t)}$$

$$\Rightarrow \frac{d^2T}{d\eta^2} = \frac{1}{\alpha}\frac{dT}{d\eta}\frac{-x(4\alpha t)}{2t(4\alpha t)^{0.5}} = -\frac{dT}{d\eta}2\eta$$

- Thus we get an ODE in η $\quad \frac{d^2T}{d\eta^2} = -2\eta\frac{dT}{d\eta}$
- The boundary conditions

$T(t,0) = T_s \Rightarrow T(\eta = 0) = T_s$

$T(0,x) = T_i$; $T(t,x\to\infty) = T_i \Rightarrow T(\eta = \infty) = T_i$

1-D Transient in a Semi-Infinite Plate-V

- The governing equation is $\frac{d^2T}{d\eta^2} = -2\eta\frac{dT}{d\eta}$
- To get the solution, we make the transformation

$$\frac{dT}{d\eta} = T^+ \Rightarrow \frac{d^2T}{d\eta^2} = \frac{dT^+}{d\eta}$$

- The equation in the new variables can be written as

$$\frac{dT^+}{d\eta} = -2\eta T^+ \qquad \Rightarrow \frac{dT^+}{T^+} = -2\eta d\eta$$

- Integration gives $\ln(T^+) = -\eta^2 + C \qquad \Rightarrow T^+ = C_1 e^{-\eta^2}$

$$\Rightarrow \frac{dT}{d\eta} = C_1 e^{-\eta^2} \Rightarrow T = C_1\int_0^\eta e^{-\eta^2}d\eta + C_2 = C_1\int_0^\eta e^{-u^2}du + C_2$$

u is a dummy variable

Error Function

- The integral $\int_0^\eta e^{-u^2}du$ occurs very frequently in physics
- Though not integrable in explicit form, it has been integrated with series expansion and tables have been constructed under what is called Error Function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

- It turns out that $\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}$
- Hence erf(∞)=1, erf(0) = 0
- A complimentary error function is also defined as

erfc(x) = 1 – erf(x)

erf(x) is tabulated in Appendix B of your book

x=3 is as good as ∞

Erf(3) = 0.99998

Integration

Consider $I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy \quad = \int_0^\infty\int_0^{2\pi} e^{-(r^2)}rdrd\theta$

Put r² = t $\Rightarrow I = \int_0^\infty\int_0^{2\pi} e^{-t}\frac{dt}{2}d\theta = \frac{2\pi}{2}\left(-e^{-t}\Big|_0^\infty\right) = \pi$

$$I = \int_{-\infty}^{\infty} e^{-x^2}dx\int_{-\infty}^{\infty} e^{-y^2}dy = 4I_1^2, \quad \text{where } I_1 = \int_0^\infty e^{-x^2}dx$$

From above $4I_1^2 = \pi$ or $I_1 = \frac{\sqrt{\pi}}{2}$

Error function can be computed by using the fact that

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots$$

1-D Transient in a Semi-Infinite Plate-VI

- The solution $T = C_1\int_0^\eta e^{-u^2}du + C_2 \qquad = C_1\frac{\sqrt{\pi}}{2}\mathrm{erf}(\eta) + C_2$
- The Boundary condition, T(η=0) = T_s implies

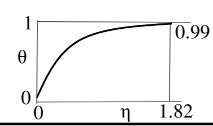
$$T_s = C_1\frac{\sqrt{\pi}}{2}\mathrm{erf}(0) + C_2 \qquad \Rightarrow C_2 = T_s$$

- The Boundary condition, T(η=∞) = T_i implies

$$T_i = C_1\frac{\sqrt{\pi}}{2}\mathrm{erf}(\infty) + T_s \quad T_i = C_1\frac{\sqrt{\pi}}{2} + T_s \quad \Rightarrow C_1 = (T_i - T_s)\frac{2}{\sqrt{\pi}}$$

$$\therefore T = (T_i - T_s)\mathrm{erf}(\eta) + T_s$$

$$\Rightarrow \frac{T - T_s}{(T_i - T_s)} = \mathrm{erf}(\eta) = \theta$$



1-D Transient in a Semi-Infinite Plate-VII

- For η = 1.82, θ=0.99; This implies that η = 1.82 is for all practical purposes is ∞

$$\eta = 1.82 \Rightarrow \frac{x}{(4\alpha t)^{0.5}} = 1.82 \qquad \Rightarrow x = 3.64(\alpha t)^{0.5}$$

- The above numbers can be interpreted in the following manner
- x > 3.64 $(\alpha t)^{0.5}$ can be considered as infinitely thick
- Similarly for t < $x^2/(13.25\ \alpha)$, the plate can be considered infinite
- Now we will turn our attention to heat transferred

$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{1}{(4\alpha t)^{0.5}}$

1-D Transient in a Semi-Infinite Plate-VIII

$$q'' = -k\frac{\partial T}{\partial x}\bigg|_{x=0} = -k\frac{dT}{d\eta}\bigg|_{\eta=0}\frac{1}{(4\alpha t)^{0.5}}$$

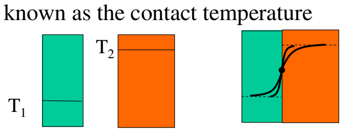
- We had shown that $\frac{dT}{d\eta} = C_1 e^{-\eta^2}$ $\Rightarrow \frac{dT}{d\eta}\Big|_{\eta=0} = C_1 = (T_i - T_s)\frac{2}{\sqrt{\pi}}$

$$\therefore q'' = -k\frac{(T_i - T_s)}{(4\alpha t)^{0.5}}\frac{2}{\sqrt{\pi}} = -k\frac{(T_s - T_i)}{(\pi\alpha t)^{0.5}}$$

- Solutions are available for other boundary conditions and are summarized in your book

Application-I

- Semi-infinite wall finds lots of applications, such as bubble growth, vapor explosion etc.
- In vapor explosion, analysis liquid metal drops falling in relatively cold water is required
- This can be idealized by two semi-infinite slabs brought into intimate contact
- We shall see the gist by trying to calculate what is known as the contact temperature



Application-II

- At the instant they come in contact, thermal equilibrium would dictate that the contact temperature T_s would lie in between T_1 and T_2
- If the contact temperature remains same, thermal equilibrium demands that the heat flux from one face should be equal to the heat flux from the other face.
- The solution on the two slabs can now be worked similar to what we have obtained with T_1 and T_2 similar to T_i

$$\therefore q_1'' = -k_1 \frac{(T_s - T_1)}{(\pi\alpha_1 t)^{0.5}} \quad \text{and} \quad q_2'' = -k_{21} \frac{(T_s - T_2)}{(\pi\alpha_2 t)^{0.5}}$$

Application-III

- Since heat flux coming out from 1 will be getting into the other,

$$\Rightarrow q_1'' = -q_2'' \qquad \Rightarrow -k_1 \frac{(T_s - T_1)}{(\pi\alpha_1 t)^{0.5}} = k_2 \frac{(T_s - T_2)}{(\pi\alpha_2 t)^{0.5}}$$

$$\Rightarrow \frac{(T_1 - T_s)}{(T_s - T_2)} = \frac{k_2}{k_1} \frac{\alpha_1^{0.5}}{\alpha_2^{0.5}} = \frac{(k_2\rho_2 c_2)^{0.5}}{(k_1\rho_1 c_1)^{0.5}}$$

- Rearranging, we get

$$T_S == \frac{T_1 (k_1\rho_1 c_1)^{0.5} + T_2 (k_2\rho_2 c_2)^{0.5}}{(k_1\rho_1 c_1)^{0.5} + (k_2\rho_2 c_2)^{0.5}}$$

Finite Difference Method - I

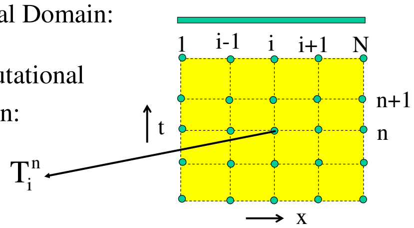
- We found that analytical methods are complex
- These methods are restrictive and are applicable for simple boundary conditions
- Numerical methods are easy to implement and we can obtain results quickly
- We will see the gist of the method

Finite Difference Method - II

Governing Equation: $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

Physical Domain:

Computational Domain:



Finite Difference Method - III

- One of the FDM approximation is

$$\left.\frac{\partial T}{\partial t}\right|_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} \qquad \left.\frac{\partial^2 T}{\partial x^2}\right|_i^n = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$

- This leads to the nodal equation

$$T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{\Delta x^2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n)$$

- This method is called explicit method, as the values at T_i^{n+1} are readily obtained explicitly, once the initial and boundary conditions are known

Finite Difference Method - IV

- This method suffers from a disadvantage that the time step $\Delta t < 0.5\alpha\Delta t/\Delta x^2$
- This is called stability limit. This occurs due to explosion of errors above the limit
- If we need more accurate results, we need more nodes, and this implies small Δx. This will limt Δt to be small and takes more computational time
- This can be overcome by choosing a different method called implicit method

Finite Difference Method - V

Implicit Method

$$\left.\frac{\partial T}{\partial t}\right|_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} \qquad \left.\frac{\partial^2 T}{\partial x^2}\right|_i^n = \frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{\Delta x^2}$$

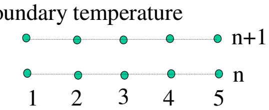
- This leads to the nodal equation

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{\Delta x^2}$$

$$T_{i+1}^{n+1}\left(-\frac{\alpha\Delta t}{\Delta x^2}\right) + T_i^{n+1}\left(1 + \frac{2\alpha\Delta t}{\Delta x^2}\right) + T_{i-1}^{n+1}\left(-\frac{\alpha\Delta t}{\Delta x^2}\right) = T_i^n$$

Finite Difference Method - VI

- For the simple case of boundary temperature known



$$\begin{bmatrix} 1 & 0 & & & \\ -\frac{\alpha\Delta t}{\Delta x^2} & 1+\frac{2\alpha\Delta t}{\Delta x^2} & -\frac{\alpha\Delta t}{\Delta x^2} & & \\ & -\frac{\alpha\Delta t}{\Delta x^2} & 1+\frac{2\alpha\Delta t}{\Delta x^2} & -\frac{\alpha\Delta t}{\Delta x^2} & \\ & & -\frac{\alpha\Delta t}{\Delta x^2} & 1+\frac{2\alpha\Delta t}{\Delta x^2} & -\frac{\alpha\Delta t}{\Delta x^2} \\ & & & 0 & 1 \end{bmatrix} \begin{Bmatrix} T_1^{n+1} \\ T_2^{n+1} \\ T_3^{n+1} \\ T_4^{n+1} \\ T_5^{n+1} \end{Bmatrix} = \begin{Bmatrix} T_1^{n+1} \\ T_3^{n} \\ T_3^{n} \\ T_4^{n} \\ T_5^{n+1} \end{Bmatrix}$$

- The matrix can be inverted in Matlab or any other software