Introduction to Convection-I

- We had introduced the concept of convection through the empirical law called the Newton's law of cooling
- We had defined the term heat transfer coefficient as

$$q'' = h(T_s - T_\infty)$$

- Till now we have been specifying this parameter in problems. Now we shall look at ways and means of computing this parameter
- Prior to going into details, we shall have a cursory look of convection and shall try to unify the subject of heat transfer to mass transfer and momentum transfer (fluid mechanics)

Introduction to Convection-II

- By mass transfer we imply the movement of one specie from one media into a mixture
- To give you a concrete example, we shall look at motion of water vapor from a pond into atmosphere of air and water vapor mixture
- Applications for these are in drying of substances, evaporative cooling, such as in cooling towers
- The unification is possible due to similar nature of governing equations and boundary conditions
- Prior to the derivation of governing equations, it will be useful to get the nomenclature sorted out in the study of mass transfer

Introduction to Convection-III

- In convective heat transfer, we just introduced

$$q'' = h(T_s - T_\infty)$$

Heat Flux (W/m²)

Heat Transfer Coefficient (W/m²-K)

- Analogously, we shall introduce the empirical equation for mass transfer as

$$N''_A = h_m(C_{As} - C_{A\infty})$$

Mass Flux of Specie-A (kmol/m²-s)

Mass Transfer Coefficient (m/s)

Concentration (kmol/m³)

Introduction to Convection-IV

- Often in mass transfer, the equations are expressed in mass basis

$$n''_A = h_m(\rho_{As} - \rho_{A\infty})$$

Mass Flux of Specie-A (kg/m²-s)

Mass Transfer Coefficient (m/s)

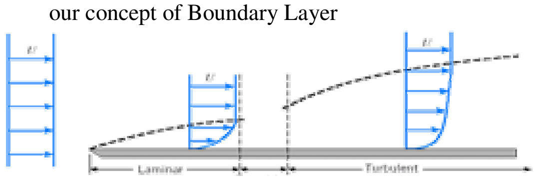
Density (kg/m³)

- Note that the above equation is obtained from the molar basis equation by multiplying both sides of it by Molecular weight

$$n'' = N''M \quad \text{and} \quad \rho = CM$$

Analogy between Mass, Momentum and Energy Transfer-I

- Consider wind blowing on a flat plate. Let us recall our concept of Boundary Layer

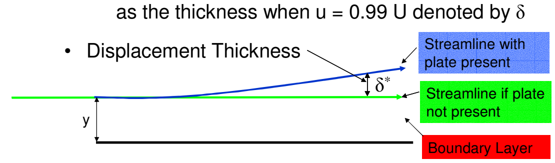


- In 1904 Prandtl introduced this concept and is considered a big milestone in Fluid mechanics
- The layer in which the velocity gradients are confined is called the boundary layer

Analogy between Mass, Momentum and Energy Transfer-II

Some Definitions

- Boundary Layer Thickness
  - Theoretically u=U only at y = ∞
  - It is customary to define boundary layer thickness as the thickness when u = 0.99 U denoted by δ
- Displacement Thickness



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Analogy between Mass, Momentum and Energy Transfer-III

- We had defined Friction Coefficient C_f as

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2}$$

- From the definition of Newton's law of Viscosity

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

- Therefore $C_f = \dfrac{\mu \left.\frac{\partial u}{\partial y}\right|_{y=0}}{\frac{1}{2}\rho u_\infty^2} = \dfrac{\mu \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} \frac{u_\infty}{L}}{\frac{1}{2}\rho u_\infty^2} = \left.\dfrac{\partial u^*}{\partial y^*}\right|_{y^*=0} \dfrac{2}{Re_L}$

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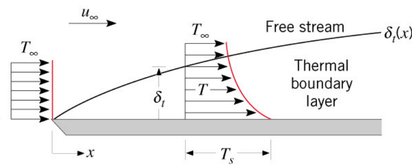
Analogy between Mass, Momentum and Energy Transfer-IV

- Analogously we can define the terms for cold air blowing on a hot plate



$$q'' = -k\left.\frac{\partial T}{\partial y}\right|_{y=0} = h(T_s - T_\infty) \qquad \Rightarrow h = -\frac{k\left.\frac{\partial T}{\partial y}\right|_{y=0}}{(T_s - T_\infty)}$$

Fourier's law

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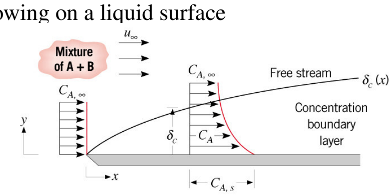
Analogy between Mass, Momentum and Energy Transfer-V

- Analogously we can define the terms for moist air blowing on a liquid surface



$$N''_A = -D_{AB}\left.\frac{\partial C_A}{\partial y}\right|_{y=0} = h_m(C_{As} - C_{A\infty}) \qquad \Rightarrow h_m = -\frac{D_{AB}\left.\frac{\partial C_A}{\partial y}\right|_{y=0}}{(C_{As} - C_{A\infty})}$$

Fick's law

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Analogy between Mass, Momentum and Energy Transfer-VI

- Usually, it is assumed that on the plate, saturation conditions exist

$$\Rightarrow C_{As} = -\frac{p_{sat}(T_s)}{R_u T_s} \quad \text{Or} \quad \rho_{As} = -\frac{p_{sat}(T_s)}{RT_s}$$

- The free stream concentration or density can be calculated from relative humidity

$$\Rightarrow \phi_{A\infty} = -\frac{p_{A\infty,}}{p_{A,sat}(T_\infty)} = \frac{\rho_{A\infty} RT_\infty}{\rho_{A,sat} RT_\infty} = \frac{\rho_{A\infty}}{\rho_{A,sat}} = \frac{C_{A\infty}}{C_{A,sat}(T_\infty)}$$

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Analogy between Mass, Momentum and Energy Transfer-VIII

- We can define non-dimensional parameters as

$$\Rightarrow h = -\frac{k\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} \frac{(T_\infty - T_s)}{L}}{(T_s - T_\infty)} \qquad \Rightarrow \frac{hL}{k} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = Nu$$

- Nu is called Nusselt Number. Note that this different from Biot number

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Analogy between Mass, Momentum and Energy Transfer-VII

- Analogously, we can define non-dimensional parameters for the case of mass transfer as

$$C_A^* = \frac{C_A - C_{As}}{C_{A\infty} - C_{As}}; \quad y^* = \frac{y}{L}$$

$$\Rightarrow h_m = -\frac{D_{AB}\left.\frac{\partial C_A^*}{\partial y^*}\right|_{y^*=0} \frac{(C_{As\infty} - C_{As})}{L}}{(C_{As} - C_{As\infty})} \qquad \Rightarrow \frac{h_m L}{k} = \left.\frac{\partial C_A^*}{\partial y^*}\right|_{y^*=0} = Sh$$

- Sh is called Sherwood Number.

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Recap of Differential Analysis - I

- As stated earlier, Heat equation is the first law of thermodynamics with only conduction
- We had derived the conservation of mass and momentum in ME 203
- These were the celebrated Navier-Stokes Equations
- Now we shall extend the concept and derive the Energy Equation
- This is perhaps the most difficult equation in terms of the number of terms

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Recap of Differential Analysis - II

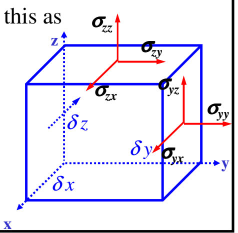
- Conservation of mass in Cartesian coordinates was derived earlier as

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

- In vector form, we could write this as

$$\frac{\partial \rho}{\partial t}+\nabla.\left(\rho\vec{V}\right)=0$$

- The state of the stress of a fluid element could be represented as shown in the figure



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Recap of Differential Analysis - III

- The momentum balance equation is a vector equation and it was derived as

$$\rho\left(\frac{\partial(u)}{\partial t}+\left(\vec{V}.\nabla\right)(u)\right)=\left(\rho g_x+\frac{\partial(\sigma_{xx})}{\partial x}+\frac{\partial(\sigma_{yx})}{\partial y}+\frac{\partial(\sigma_{zx})}{\partial z}\right)$$

$$\rho\left(\frac{\partial(v)}{\partial t}+\left(\vec{V}.\nabla\right)(v)\right)=\left(\rho g_y+\frac{\partial(\sigma_{xy})}{\partial x}+\frac{\partial(\sigma_{yy})}{\partial y}+\frac{\partial(\sigma_{zy})}{\partial z}\right)$$

$$\rho\left(\frac{\partial(w)}{\partial t}+\left(\vec{V}.\nabla\right)(w)\right)=\left(\rho g_z+\frac{\partial(\sigma_{xz})}{\partial x}+\frac{\partial(\sigma_{yz})}{\partial y}+\frac{\partial(\sigma_{zz})}{\partial z}\right)$$

- This equation has density, 3 velocities and 9 stresses.
- To close the system, Generalized Newtonian Stress-Strain relations were introduced without proof

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Recap of Differential Analysis - IV

$$\sigma_{xx}=-p+2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left(\nabla.\vec{V}\right)$$

$$\sigma_{yy}=-p+2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\left(\nabla.\vec{V}\right)$$

$$\sigma_{zz}=-p+2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu\left(\nabla.\vec{V}\right)$$

Normal stresses

$$\sigma_{xy}=\sigma_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\sigma_{yz}=\sigma_{zy}=\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

$$\sigma_{zx}=\sigma_{xz}=\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

Shear stresses

We have all stresses in terms of pressure and velocities

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Recap of Differential Analysis - V

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]+\frac{\partial}{\partial x}\left[\frac{\mu}{3}\left(\nabla.\vec{V}\right)\right]+\rho g_x$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]+\frac{\partial}{\partial y}\left[\frac{\mu}{3}\left(\nabla.\vec{V}\right)\right]+\rho g_y$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]+\frac{\partial}{\partial z}\left[\frac{\mu}{3}\left(\nabla.\vec{V}\right)\right]+\rho g_z$$

$$\rho\frac{D\vec{V}}{Dt}=-\nabla P+\mu\nabla^2\vec{V}+\frac{\mu}{3}\nabla\left(\nabla.\vec{V}\right)+\rho\vec{g}$$

- The only assumption is the equation for stresses, which is the generalized law for Newtonian Fluids

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Conservation of Energy-I

- Previously we derived the energy equation (First law of Thermodynamics) with only conduction
- Now we will modify it with Convection or Fluid Motion
- Conceptually stated for a control volume,

Rate of increase of energy in the control volume	=	**Net Rate of energy flowing into the control volume by conduction and convection**	+	**Rate of energy generated within the control volume**

$$\dot{E}_{CV}=\dot{E}_{net}+\dot{E}_{gen}$$

Conservation of Energy-II

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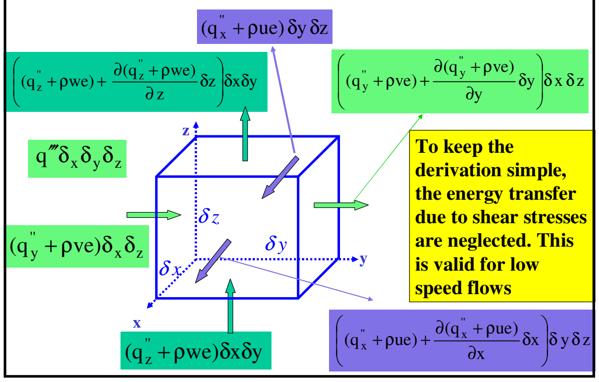

Conservation of Energy-III

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In our derivation for gases $e = c_v T + \frac{u^2 + v^2 + w^2}{2} + gH$

- Looking at the net rate of energy convecting and diffusing in, we can write

$$\dot{E}_{net} = \dot{E}_{in} - \dot{E}_{out}$$

$$\dot{E}_{in} = (q_x'' + \rho ue)\,\delta y\,\delta z + (q_y'' + \rho ve)\,\delta x\,\delta z + (q_z'' + \rho we)\,\delta x\,\delta y$$

$$\dot{E}_{out} = \left((q_x'' + \rho ue) + \frac{\partial (q_x'' + \rho ue)}{\partial x}\delta x\right)\delta y\,\delta z + \left((q_y'' + \rho ve) + \frac{\partial (q_y'' + \rho ve)}{\partial y}\delta y\right)\delta x\,\delta z + \left((q_z'' + \rho we) + \frac{\partial (q_w'' + \rho we)}{\partial z}\delta z\right)\delta x\,\delta y$$

Conservation of Energy-IV

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- Looking at the net rate of energy convecting and diffusing in, we can write

$$\dot{E}_{net} = -\left(\frac{\partial (q_x'' + \rho ue)}{\partial x} + \frac{\partial (q_y'' + \rho ve)}{\partial y} + \frac{\partial (q_w'' + \rho we)}{\partial z}\right)\delta x\,\delta y\,\delta z$$

- The change of energy in the control volume can be expressed as

$$\dot{E}_{CV} = \frac{\partial (\delta_x \delta_y \delta_z \rho c_v T)}{\partial t}$$

- Rate of heat generated in control volume is equal to

$$\dot{E}_{gen} = q'''\delta x \delta y \delta z$$

Conservation of Energy-V

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$$\dot{E}_{CV} = \dot{E}_{net} + \dot{E}_{gen}$$

$$\frac{\partial (\rho c_v T)}{\partial t}\delta x \delta y \delta z = q'''\delta x \delta y \delta z - \left(\frac{\partial (q_x'' + \rho ue)}{\partial x} + \frac{\partial (q_y'' + \rho ve)}{\partial y} + \frac{\partial (q_w'' + \rho we)}{\partial z}\right)\delta x\,\delta y\,\delta z$$

- In steady state

$$-\left(\frac{\partial (q_x'' + \rho ue)}{\partial x} + \frac{\partial (q_y'' + \rho ve)}{\partial y} + \frac{\partial (q_w'' + \rho we)}{\partial z}\right) + q''' = 0$$

$$q_x'' = -k\frac{\partial T}{\partial x}; \quad q_y'' = -k\frac{\partial T}{\partial y}; \quad q_z'' = -k\frac{\partial T}{\partial z}; \quad e = c_v T$$

Conservation of Energy-VI

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- Thus, the energy equation can be written as

$$\frac{\partial (\rho u c_v T)}{\partial x} + \frac{\partial (\rho v c_v T)}{\partial y} + \frac{\partial (\rho w c_v T)}{\partial z} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q'''$$

- This is the conservative form of energy equation for low speed flows
- The left hand side can be simplified as

$$\rho u \frac{\partial (c_v T)}{\partial x} + c_v T \frac{\partial (\rho u)}{\partial x} + \rho v \frac{\partial (c_v T)}{\partial y} + c_v T \frac{\partial (\rho v)}{\partial y} + \rho w \frac{\partial (c_v T)}{\partial z} + c_v T \frac{\partial (\rho w)}{\partial z}$$

$$= c_v T\left(\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z}\right) + \rho u \frac{\partial (c_v T)}{\partial x} + \rho v \frac{\partial (c_v T)}{\partial y} + \rho w \frac{\partial (c_v T)}{\partial z}$$

$$= \rho u \frac{\partial (c_v T)}{\partial x} + \rho v \frac{\partial (c_v T)}{\partial y} + \rho w \frac{\partial (c_v T)}{\partial z}$$

Conservation of Energy-VII

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- If property c_v can be treated as constant, we can write

$$\rho u c_v \frac{\partial (T)}{\partial x} + \rho v c_v \frac{\partial (T)}{\partial y} + \rho w c_v \frac{\partial (T)}{\partial z} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q'''$$

$$u \frac{\partial (T)}{\partial x} + v \frac{\partial (T)}{\partial y} + w \frac{\partial (T)}{\partial z} = \frac{k}{\rho c_v}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q'''$$

- If the fluid is incompressible, $c_v = c_p = c$

$$\rho c\left(u \frac{\partial (T)}{\partial x} + v \frac{\partial (T)}{\partial y} + w \frac{\partial (T)}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q'''$$

- The more accurate form of the equation by taking pressure work into account can be written as

$$\rho c_p\left(u \frac{\partial (T)}{\partial x} + v \frac{\partial (T)}{\partial y} + w \frac{\partial (T)}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q'''$$

Conservation of Species-I

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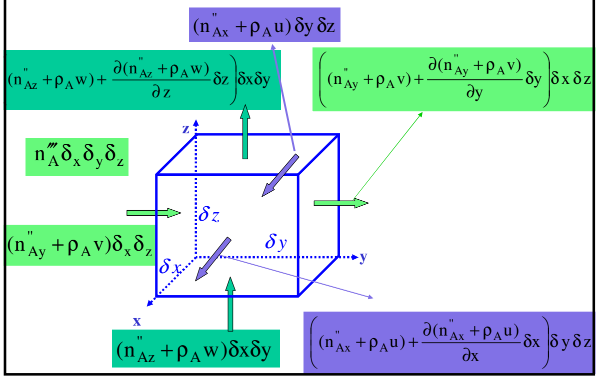

Conservation of Species-II

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$$\frac{\partial(M_{A-CV})}{\partial t}=\dot{M}_{A-net}+\dot{M}_{gen}$$

$$\frac{\partial(\rho_A)}{\partial t}\delta x\delta y\delta z=n'''\delta x\delta y\delta z$$

$$-\left(\frac{\partial(n''_{Ax}+\rho_A u)}{\partial x}+\frac{\partial(n''_{Ay}+\rho_A v)}{\partial y}+\frac{\partial(n''_{Aw}+\rho_A w)}{\partial z}\right)\delta x\delta y\,\delta z$$

- In steady state

$$-\left(\frac{\partial(n''_{Ax}+\rho_A u)}{\partial x}+\frac{\partial(n''_{Ay}+\rho_A v)}{\partial y}+\frac{\partial(n''_{Aw}+\rho_A w)}{\partial z}\right)+n'''_A=0$$

$$n''_{Ax}=-D_{AB}\frac{\partial\rho_A}{\partial x};\quad n''_y=-D_{AB}\frac{\partial\rho_A}{\partial y};\quad n''_z=-D_{AB}\frac{\partial\rho_A}{\partial z};$$

Conservation of Species-III

27

- Thus, the species conservation equation can be written as

$$\frac{\partial(\rho_A u)}{\partial x}+\frac{\partial(\rho_A v)}{\partial y}+\frac{\partial(\rho_A w)}{\partial z}=D_{AB}\left(\frac{\partial^2\rho_A}{\partial x^2}+\frac{\partial^2\rho_A}{\partial y^2}+\frac{\partial^2\rho_A}{\partial z^2}\right)+n'''_A$$

- This is the conservative form of species conservation equation
- The left hand side can be simplified as

$$u\frac{\partial(\rho_A)}{\partial x}+\rho_A\frac{\partial(u)}{\partial x}+v\frac{\partial(\rho_A)}{\partial y}+\rho_A\frac{\partial(v)}{\partial y}+w\frac{\partial(\rho_A)}{\partial z}+\rho_A\frac{\partial(w)}{\partial z}$$

$$=\rho_A\left(\frac{\partial(u)}{\partial x}+\frac{\partial(v)}{\partial y}+\frac{\partial(w)}{\partial z}\right)+u\frac{\partial(\rho_A)}{\partial x}+v\frac{\partial(\rho_A)}{\partial y}+\rho\frac{\partial(\rho_A)}{\partial z}$$

For incompressible flows

$$=u\frac{\partial(\rho_A)}{\partial x}+v\frac{\partial(\rho_A)}{\partial y}+w\frac{\partial(\rho_A)}{\partial z}$$

$$\therefore\; u\frac{\partial(\rho_A)}{\partial x}+v\frac{\partial(\rho_A)}{\partial y}+w\frac{\partial(\rho_A)}{\partial z}=D_{AB}\left(\frac{\partial^2\rho_A}{\partial x^2}+\frac{\partial^2\rho_A}{\partial y^2}+\frac{\partial^2\rho_A}{\partial z^2}\right)+n'''_A$$

Momentum, Heat and Mass Transfer Analogy - I

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- The simplified momentum, energy and species conservation equations for incompressible flow with no body force can be summarized as

$$u\frac{\partial(u)}{\partial x}+v\frac{\partial(u)}{\partial y}+w\frac{\partial(u)}{\partial z}=\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\partial p}{\partial x}$$

$$u\frac{\partial(v)}{\partial x}+v\frac{\partial(v)}{\partial y}+w\frac{\partial(v)}{\partial z}=\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\partial p}{\partial y}$$

$$u\frac{\partial(w)}{\partial x}+v\frac{\partial(w)}{\partial y}+w\frac{\partial(w)}{\partial z}=\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\frac{\partial p}{\partial z}$$

$$\left(u\frac{\partial(T)}{\partial x}+v\frac{\partial(T)}{\partial y}+w\frac{\partial(T)}{\partial z}\right)=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+q'''$$

$$u\frac{\partial(\rho_A)}{\partial x}+v\frac{\partial(\rho_A)}{\partial y}+w\frac{\partial(\rho_A)}{\partial z}=D_{AB}\left(\frac{\partial^2\rho_A}{\partial x^2}+\frac{\partial^2\rho_A}{\partial y^2}+\frac{\partial^2\rho_A}{\partial z^2}\right)+n'''_A$$

Momentum, Heat and Mass Transfer Analogy - II

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- In case of two-dimensional boundary layer flows over a flat plate with no source terms, we can simplify the equations as

$$u\frac{\partial(u)}{\partial x}+v\frac{\partial(u)}{\partial y}=\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\partial p}{\partial x}$$

$$\left(u\frac{\partial(T)}{\partial x}+v\frac{\partial(T)}{\partial y}\right)=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+q'''$$

$$u\frac{\partial(\rho_A)}{\partial x}+v\frac{\partial(\rho_A)}{\partial y}=D_{AB}\left(\frac{\partial^2\rho_A}{\partial x^2}+\frac{\partial^2\rho_A}{\partial y^2}\right)+n'''_A$$

- Introducing

$$u^*=\frac{u}{u_\infty};\quad v^*=\frac{v}{u_\infty}\quad T^*=\frac{T-T_s}{T_\infty-T_s};\quad \rho_A{}^*=\frac{\rho-\rho_{As}}{\rho_{A\infty}-\rho_{As}};\quad x^*=\frac{x}{L};\quad y^*=\frac{y}{L}$$

Momentum, Heat and Mass Transfer Analogy - III

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- The non-dimensional equations can be written as

$$u^*\frac{\partial(u^*)}{\partial x^*}+v\frac{\partial(u^*)}{\partial y^*}=\frac{\mu}{\rho u_\infty L}\left(\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$\left(u^*\frac{\partial(T^*)}{\partial x^*}+v^*\frac{\partial(T^*)}{\partial y^*}\right)=\frac{k}{\rho c_p u_\infty L}\left(\frac{\partial^2 T^*}{\partial y^{*2}}\right)$$

$$u^*\frac{\partial(\rho_A{}^*)}{\partial x^*}+v^*\frac{\partial(\rho_A{}^*)}{\partial y^*}=\frac{D_{AB}}{u_\infty L}\left(\frac{\partial^2\rho_A{}^*}{\partial x^{*2}}+\frac{\partial^2\rho_A{}^*}{\partial y^{*2}}\right)$$

- The Parameters on right are defined as follows

$$\frac{1}{Re_L}=\frac{\mu}{\rho u_\infty L};\qquad \frac{1}{Re_L\,Pr}=\frac{k}{\rho c_p u_\infty L};\qquad \frac{1}{Re_L\,Sc}=\frac{D_{AB}}{u_\infty L};$$

where, $Pr=\frac{\nu}{\alpha}$; $\quad Sc=\frac{\nu}{D_{AB}}$ Prandtl Number, Schmidt Number

Momentum, Heat and Mass Transfer Analogy - IV

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- The final form of non-dimensional equations can be written as

$$u^*\frac{\partial(u^*)}{\partial x^*}+v\frac{\partial(u^*)}{\partial y^*}=\frac{1}{\text{Re}_L}\left(\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$\left(u^*\frac{\partial(T^*)}{\partial x^*}+v^*\frac{\partial(T^*)}{\partial y^*}\right)=\frac{1}{\text{Re}_L\,\text{Pr}}\left(\frac{\partial^2 T^*}{\partial y^{*2}}\right)$$

$$u^*\frac{\partial(\rho_A^*)}{\partial x^*}+v^*\frac{\partial(\rho_A^*)}{\partial y^*}=\frac{1}{\text{Re}_L\,\text{Sc}}\left(\frac{\partial^2 \rho_A^*}{\partial y^{*2}}\right)$$

- Most gases have Pr and Sc approximately 1
- Hence the non-dimensional equations are identical
- The way we have defined the non-dimensional variables, the boundary conditions are also identical

Momentum, Heat and Mass Transfer Analogy - V

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- Hence the solutions for the non-dimensional variables shall also be identical
- Therefore, the slopes at the wall shall also be identical

$$\left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0}=\text{Re}_L\frac{C_f}{2}\qquad=\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}=\text{Nu}\qquad=\left.\frac{\partial \rho_A^*}{\partial y^*}\right|_{y^*=0}=\text{Sh}$$

$$\Rightarrow \text{Re}_L\frac{C_f}{2}=\text{Nu}=\text{Sh}$$

Momentum, Heat and Mass Transfer Analogy - VI

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- From the non-dimensional equations, we can also deduce the following functional form

$$C_f=f(\text{Re}_L)\qquad \text{Nu}=f(\text{Re}_L,\text{Pr})\qquad \text{Sh}=f(\text{Re}_L,\text{Sc})$$

- For Pr = 1

$$\delta=\delta_T=\delta_m$$

- Since Pr = ν/α, it represents the ratio of momentum diffusivity to thermal diffusivity. Hence,

$$\text{For}\quad \text{Pr}>1;\ \ \delta>\delta_T$$

- Similarly, Sc = ν/D_{AB}, it represents the ratio of momentum diffusivity to mass diffusivity. Hence,

$$\text{For}\quad \text{Sc}>1;\ \ \delta>\delta_m$$

Momentum, Heat and Mass Transfer Analogy - VII

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- We also make the following observation from intuition, which will be proved later
- For Pr not equal to 1

$$\frac{\delta}{\delta_T}=\text{Pr}^n$$

- For Sc not equal to 1

$$\frac{\delta}{\delta_m}=\text{Sc}^n$$

- From the above

$$\frac{\delta_T}{\delta_m}=\left(\frac{\text{Sc}}{\text{Pr}}\right)^n=\text{Le}^n\qquad \text{where}\ \ \text{Le}=\frac{\text{Sc}}{\text{Pr}}=\frac{\alpha}{D_{AB}}$$

For laminar flows, n = 1/3

Lewis Number

Momentum, Heat and Mass Transfer Analogy - VII

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- Since $\text{Nu}=f(\text{Re}_L,\text{Pr})$ $\quad \text{Sh}=f(\text{Re}_L,\text{Sc})$
- We can write the relationship as a power law

$$\text{Nu}=\frac{hL}{k}=C\ \text{Re}_L^{\,m}\,\text{Pr}^n\qquad \text{Sh}=\frac{h_m L}{D_{AB}}=C\ \text{Re}_L^{\,m}\,\text{Sc}^n$$

- This implies that

$$\frac{h}{h_m}=\frac{k}{D_{AB}}\frac{\text{Pr}^n}{\text{Sc}^n}=\frac{\alpha\rho c_p}{D_{AB}}\frac{\text{Pr}^n}{\text{Sc}^n}=\rho c_p\frac{\text{Le}}{(\text{Le})^n}=\rho c_p(\text{Le})^{1-n}$$

$$\Rightarrow \frac{h}{h_m}=\rho c_p(\text{Le})^{1-n}$$

Remember that the density and specific heat is that of mixture

Application of Heat and Mass Transfer Analogy - I

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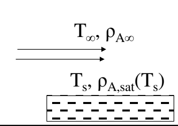
- The most common application of this analogy is the evaporative cooling
- It is commonly adopted in cooling towers where hot water is allowed to fall in the form of a sprays from top and relatively dry air is blown from below to create evaporation and thereby extract the latent heat from the spray and cool.
- In case of steady evaporative cooling, the heat convected from air into the cooled liquid is equal to the heat required to evaporate the necessary amount of moisture for mass transfer

$$q''_{evap}=n''_{evap}\,h_{fg}=h_m\left(\rho_{A,sat}(T_s)-\rho_{A,\infty}\right)h_{fg}$$

$$q''_{conv}=h\left(T_\infty-T_s\right)$$



Application of Heat and Mass Transfer Analogy - II

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$$\Rightarrow h(T_\infty - T_s) = h_m(\rho_{A,sat}(T_s) - \rho_{A,\infty})\ h_{fg}$$

$$\Rightarrow (T_\infty - T_s) = \frac{h_m}{h}(\rho_{A,sat}(T_s) - \rho_{A,\infty})\ h_{fg}$$

$$\frac{h}{h_m} = \rho c_p (Le)^{1-n}$$

- We use the relation of h/h_m derived earlier and the equation of state to give

$$\Rightarrow (T_\infty - T_s) = \frac{1}{\rho c_p Le^{1-n}} \frac{\left(\frac{p_{A,sat}(T_s)}{T_s} - \frac{p_{A,\infty}}{T_\infty}\right)}{R_u / M_A}\ h_{fg}$$

$$\Rightarrow (T_\infty - T_s) \approx \frac{1}{\rho c_p Le^{1-n}} \frac{(p_{A,sat}(T_s) - p_{A,\infty})}{\overline{T}(R_u / M_A)}\ h_{fg}$$

Application of Heat and Mass Transfer Analogy - III

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- The mean temperature is then eliminated as follows
- In most practical cooling problem, the mass fraction of vapor is very small and hence density of mixture is approximately equal to density of component B

Hence $\rho \overline{T} = \frac{p}{R_u / M_B}$

$$\Rightarrow (T_\infty - T_s) \approx \frac{M_A}{M_B} \frac{h_{fg}}{\rho c_p Le^{1-n}} \left(\frac{p_{A,sat}(T_s) - p_{A,\infty}}{p}\right)$$