Heat, Mass and Momentum Transfer Over A Flat Plate - I

- The final form Boundary layer equations Over a flat plate was derived as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial (u)}{\partial x}+v\frac{\partial (u)}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\left(u\frac{\partial (T)}{\partial x}+v\frac{\partial (T)}{\partial y}\right)=\alpha\left(\frac{\partial^2 T}{\partial y^2}\right)$$

$$u\frac{\partial (\rho_A)}{\partial x}+v\frac{\partial (\rho_A)}{\partial y}=D_{AB}\left(\frac{\partial^2 \rho_A}{\partial y^2}\right)$$

- The exact solution can be obtained using similarity solution

Heat, Mass and Momentum Transfer Over A Flat Plate - II

- First we shall derive the governing equations for momentum transfer
- We had seen the concept of Stream Function in Fluid Mechanics
- The definition of stream function is such that the velocity components are obtained through partial derivatives as given by

$$u=\frac{\partial \psi}{\partial y} \qquad v=-\frac{\partial \psi}{\partial x}$$

- The motivation for stating this way is that the stream function would automatically satisfy the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial^2 \psi}{\partial x\partial y}-\frac{\partial^2 \psi}{\partial x\partial y}=0$$

Heat, Mass and Momentum Transfer Over A Flat Plate - III

- Now a similarity variable η is defined as

$$\eta=y\sqrt{\frac{u_\infty}{\nu x}} \quad\Rightarrow\frac{\partial \eta}{\partial y}=\sqrt{\frac{u_\infty}{\nu x}}$$

$$\Rightarrow\frac{\partial \eta}{\partial x}=y\sqrt{\frac{u_\infty}{\nu}}\left(-\frac{1}{2}\right)x^{-3/2}=-\frac{1}{2}\frac{y}{x}\sqrt{\frac{u_\infty}{\nu x}}=-\frac{1}{2}\frac{\eta}{x}$$

- The stream function is now assumed to be of the form

$$\psi=f(\eta)\sqrt{u_\infty \nu x}$$

- Now we try to transform the momentum equation as follows

$$u=\frac{\partial \psi}{\partial y}=\sqrt{u_\infty \nu x}\left(\frac{df}{dn}\frac{\partial \eta}{\partial y}\right)=\sqrt{u_\infty \nu x}\left(\frac{df}{dn}\sqrt{\frac{u_\infty}{\nu x}}\right)=u_\infty\frac{df}{dn}$$

Heat, Mass and Momentum Transfer Over A Flat Plate - IV

$$\Rightarrow\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(u_\infty\frac{df}{d\eta}\right)=u_\infty\frac{d^2f}{d\eta^2}\frac{\partial \eta}{\partial x}=u_\infty\frac{d^2f}{d\eta^2}\frac{-\eta}{2x}$$

$$\Rightarrow\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(u_\infty\frac{df}{d\eta}\right)=u_\infty\frac{d^2f}{d\eta^2}\frac{\partial \eta}{\partial y}=u_\infty\frac{d^2f}{d\eta^2}\sqrt{\frac{u_\infty}{\nu x}}$$

$$\Rightarrow\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(u_\infty\frac{d^2f}{d\eta^2}\sqrt{\frac{u_\infty}{\nu x}}\right)=u_\infty\sqrt{\frac{u_\infty}{\nu x}}\frac{d^3f}{d\eta^3}\sqrt{\frac{u_\infty}{\nu x}}=u_\infty\frac{u_\infty}{\nu x}\frac{d^3f}{d\eta^3}$$

$$v=-\frac{\partial \psi}{\partial x}=-\left\{\left(\sqrt{u_\infty \nu x}\frac{df}{d\eta}\frac{\partial \eta}{\partial x}\right)+\left(f\sqrt{u_\infty \nu}\frac{1}{2\sqrt{x}}\right)\right\}$$

$$=-\left\{\left(\sqrt{u_\infty \nu x}\frac{df}{d\eta}\frac{-\eta}{2x}\right)+\left(f\sqrt{\frac{u_\infty \nu}{x}}\frac{-1}{2}\right)\right\}=\frac{1}{2}\sqrt{\frac{u_\infty \nu}{x}}\left\{\left(\eta\frac{df}{d\eta}\right)-f\right\}$$

Heat, Mass and Momentum Transfer Over A Flat Plate - V

$$\therefore\ v=\frac{1}{2}\sqrt{\frac{u_\infty \nu}{x}}\left\{\left(\eta\frac{df}{d\eta}\right)-f\right\}$$

- Substituting these in the momentum equation, we get

$$\left(u_\infty\frac{df}{dn}\right)\left(u_\infty\frac{d^2f}{d\eta^2}\frac{-\eta}{2x}\right)+\frac{1}{2}\sqrt{\frac{u_\infty \nu}{x}}\left\{\left(\eta\frac{df}{d\eta}\right)-f\right\}u_\infty\frac{d^2f}{d\eta^2}\sqrt{\frac{u_\infty}{\nu x}}$$

u $\quad\frac{\partial u}{\partial x}\quad$ v $\quad\frac{\partial v}{\partial y}$

$$=\nu u_\infty\frac{u_\infty}{\nu x}\frac{d^3f}{d\eta^3}$$

$\nu\frac{\partial^2 u}{\partial y^2}$

Collecting terms

$$-\eta\frac{u_\infty^2}{2x}\frac{df}{d\eta}\left(\frac{d^2f}{d\eta^2}\right)+\frac{u_\infty^2}{2x}\left(\eta\frac{df}{d\eta}-f\right)\frac{d^2f}{d\eta^2}=\frac{u_\infty^2}{x}\frac{d^3f}{d\eta^3}$$

$$\Rightarrow -f\frac{d^2f}{d\eta^2}\frac{u_\infty^2}{2x}=\frac{u_\infty^2}{x}\frac{d^3f}{d\eta^3}$$

$$\Rightarrow f\frac{d^2f}{d\eta^2}+2\frac{d^3f}{d\eta^3}=0$$

- The above equation was derived by Blasius and is called Blasius Equation
- The Boundary conditions

$$\therefore\ v=\frac{1}{2}\sqrt{\frac{u_\infty \nu}{x}}\left\{\left(\eta\frac{df}{d\eta}\right)-f\right\}$$

$$u=0,\ \ v=0\ \ \text{at}\ \ y=0 \quad\Rightarrow\quad \frac{df}{d\eta}=0,\ \ f=0\ \ \text{at}\ \ \eta=0$$

$$u=u_\infty,\ \ \text{at}\ \ y=\infty;\quad u=u_\infty,\ \ \text{at}\ \ x=0 \quad\Rightarrow\quad \frac{df}{d\eta}=1,\ \ \text{at}\ \ \eta=\infty$$

$$u=u_\infty\frac{df}{dn}$$

Solution of Blasius Equation-I

- Blasius equation given below does not have a closed form solution

$$\Rightarrow f\frac{d^2 f}{d\eta^2} + 2\frac{d^3 f}{d\eta^3} = 0$$

- It is easy to solve it numerically
- As the equation is third order ODE, it is usually split into three first order differential equation

$$\text{Let } f = y_1, \quad \frac{df}{d\eta} = y_2, \quad \frac{d^2 f}{d\eta^2} = y_3$$

$$\frac{dy_1}{d\eta} = y_2 \qquad y_1(0) = 0$$

$$\frac{dy_2}{d\eta} = y_3 \qquad y_2(0) = 0$$

$$\Rightarrow \frac{dy_3}{d\eta} = -\frac{y_1 y_3}{2} \qquad y_2(\infty) = 1$$

Solution of Blasius Equation-II

- There are many methods to solve. But we shall see the simplest of all called Euler's method
- This is not the most accurate method, but is easy to follow
- From Taylor series, we can write

$$y(\eta + d\eta) \approx y(\eta) + \left.\frac{dy}{d\eta}\right|_\eta \Delta\eta$$

- Now if we apply it to a simple differential equation

$$\frac{dy_1}{d\eta} = y_2$$

$$y_1(0+\Delta\eta) \approx y_1(0) + \left.\frac{dy_1}{d\eta}\right|_0 \Delta\eta \Rightarrow y_1(\Delta\eta) = y_1(0) + y_2(0)\Delta\eta$$

Solution of Blasius Equation-III

- Now if we apply it to the second equation

$$\frac{dy_2}{d\eta} = y_3$$

$$\Rightarrow y_2(\Delta\eta) = y_2(0) + y_3(0)\Delta\eta$$

- Now if we apply it to the third equation

$$\Rightarrow \frac{dy_3}{d\eta} = -\frac{y_1 y_3}{2}$$

$$\Rightarrow y_3(\Delta\eta) = y_3(0) - \frac{y_1(0)y_3(0)}{2}\Delta\eta$$

- From the boundary conditions $y_1(0)$, $y_2(0)$ are known, But $y_3(0)$ is not known, but $y_2(\infty)$ is what is known
- The way to solve these is to assume $y_3(0)$, and proceed forward and check whether $y_2(\infty) = 1$. If not take another guess and repeat

Solution of Blasius Equation-IV

- Discussion on the numerical solution

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu u_\infty \left.\frac{d^2 f}{d\eta^2}\right|_{(\eta=0)} \sqrt{\frac{u_\infty}{\nu x}} = 0.332\mu u_\infty \sqrt{\frac{u_\infty}{\nu x}}$$

$$C_f = \frac{\tau_w}{0.5\rho u_\infty^2} = \frac{0.332\mu u_\infty \sqrt{\frac{u_\infty}{\nu x}}}{0.5\rho u_\infty^2} = 0.664\sqrt{\frac{\nu}{u_\infty x}}$$

$$\therefore C_f == \frac{0.664}{\sqrt{Re_x}}$$

Solution of Blasius Equation-V

- We had shown earlier that for Flat Plate, Pr = 1 and Sc = 1, the non dimensional equations are similar
- Hence, applying analogy

$$Re\frac{C_f}{2} = Nu = Sh$$

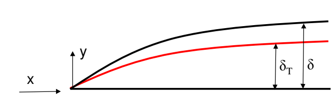
$$\Rightarrow Nu_x = Re_x \frac{0.664}{2\sqrt{Re_x}} = 0.332\sqrt{Re_x}$$

$$\Rightarrow Sh_x = Re_x \frac{0.664}{2\sqrt{Re_x}} = 0.332\sqrt{Re_x}$$

- The dependence on Prandtl number is still not visible
- We shall show this by the integral method

Integral Method-I

- In Fluid Mechanics we had seen that the solution for C_f could be obtained approximately by integral method
- We shall proceed systematically to develop this for heat transfer
- Since energy equation cannot be solved until momentum equation is solved, we have to handle both the equations
- In this treatment, we assume that $\delta_T \le \delta$, or $Pr \ge 1$



- Integrating Continuity Equation across Boundary layer

$$\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta \frac{\partial v}{\partial y}dy = 0 \qquad \Rightarrow \int_0^\delta \frac{\partial u}{\partial x}dy + v|_\delta - v|_0 = 0$$

Integral Method-II

$$\Rightarrow v|_\delta = -\int_0^\delta \frac{\partial u}{\partial x}dy \quad \text{..................} \quad (1)$$

- The momentum equation is

$$u\frac{\partial(u)}{\partial x} + v\frac{\partial(u)}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial(u^2)}{\partial x} - u\frac{\partial(u)}{\partial x} + \frac{\partial(uv)}{\partial y} - u\frac{\partial(v)}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\therefore \frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

Conservative form (2)

- Integrating Momentum Equation across Boundary layer

$$\int_0^\delta \frac{\partial(u^2)}{\partial x}dy + \int_0^\delta \frac{\partial(uv)}{\partial y}dy = \int_0^\delta \nu\left(\frac{\partial^2 u}{\partial y^2}\right)dy$$

Integral Method-III

$$\int_0^\delta \frac{\partial(u^2)}{\partial x}dy + uv|_\delta - uv|_0 = \nu\left(\frac{\partial u}{\partial y}\bigg|_\delta - \frac{\partial u}{\partial y}\bigg|_0\right)$$

$$\Rightarrow \int_0^\delta \frac{\partial(u^2)}{\partial x}dy + u_\infty\left(-\int_0^\delta \frac{\partial(u)}{\partial x}dy\right) = -\nu\frac{\partial u}{\partial y}\bigg|_0$$

From Eq. (1)

$$\Rightarrow \int_0^\delta \frac{\partial(u^2 - uu_\infty)}{\partial x}dy = -\frac{\tau_w}{\rho}$$

$$\Rightarrow \frac{d}{dx}\int_0^\delta u(u-u_\infty)dy = -\frac{\tau_w}{\rho}$$

$$\Rightarrow \frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(\frac{u}{u_\infty}-1\right)dy = -\frac{\tau_w}{\rho u_\infty^2} = -\frac{C_f}{2}$$

(3) **Integral Momentum Eq**

Integral Method-IV

- The energy equation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial(uT)}{\partial x} - T\frac{\partial(u)}{\partial x} + \frac{\partial(vT)}{\partial y} - T\frac{\partial(v)}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right)$$

$$\therefore \frac{\partial(uT)}{\partial x} + \frac{\partial(vT)}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right)$$

Conservative form (4)

- Integrating Energy Equation across Boundary layer

$$\int_0^{\delta_T} \frac{\partial(uT)}{\partial x}dy + \int_0^{\delta_T} \frac{\partial(vT)}{\partial y}dy = \int_0^{\delta_T} \alpha\left(\frac{\partial^2 T}{\partial y^2}\right)dy$$

Integral Method-V

$$\int_0^{\delta_T} \frac{\partial(uT)}{\partial x}dy + vT|_{\delta_T} - vT|_0 = \alpha\left(\frac{\partial T}{\partial y}\bigg|_{\delta_T} - \frac{\partial T}{\partial y}\bigg|_0\right)$$

$$\Rightarrow \int_0^{\delta_T} \frac{\partial(uT)}{\partial x}dy + T_\infty\left(-\int_0^{\delta_T} \frac{\partial(u)}{\partial x}dy\right) = -\alpha\frac{\partial T}{\partial y}\bigg|_0$$

From Eq. (1) → **This requires $\delta_T < \delta$**

$$\Rightarrow \int_0^{\delta_T} \frac{\partial(uT - uT_\infty)}{\partial x}dy = -\frac{q''_w}{\rho c_p}$$

$$\Rightarrow \frac{d}{dx}\int_0^{\delta_T} u(T-T_\infty)dy = -\frac{q''_w}{\rho c_p}$$

(5) **Integral Energy Eq**

Integral Method-VI

- We had just derived the integral momentum equation as

$$\Rightarrow \frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(\frac{u}{u_\infty}-1\right)dy = -\frac{\tau_w}{\rho u_\infty^2} = -\frac{C_f}{2}$$

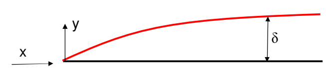
(3) **Integral Momentum Eq**

- For Evaluation of wall shear stress and hence the drag, we need to know the velocity profile
- Integral method assumes realistic profiles to get the answer
- Assume a third order polynomial

$$u = a + by + cy^2 + dy^3$$

- The constants are evaluated using boundary conditions

Integral Method-VII



$u = 0$ at $y = 0$

$u = u_\infty$ at $y = \delta$

$\frac{du}{dy} = 0$ at $y = \delta$

$\frac{d^2u}{dy^2} = 0$ at $y = 0$

- Evaluation of the constants lead to

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{y^3}{2\delta^3}$$

- Further

$$\tau_w = \mu\left(\frac{du}{dy}\right)_{y=0} = 2\mu u_\infty\left(\frac{3}{2\delta} - \frac{3y^2}{2\delta^3}\right)_{y=0} = \frac{\mu u_\infty}{\delta}\frac{3}{2}$$

Integral Method-VIII

- To simplify the algebra let us define $\eta = \frac{y}{\delta} \Rightarrow \delta d\eta = dy$

$$\frac{u}{u_\infty} = \frac{3}{2}\eta - \frac{\eta^3}{2}$$

$$\therefore \tau_w = \frac{3}{2}\frac{\mu u_\infty}{\delta} = \rho u_\infty^2 \frac{d}{dx}\left(\int_0^\delta \frac{u}{u_\infty}(1-\frac{u}{u_\infty})dy\right)$$

$$\Rightarrow \frac{3\mu u_\infty}{2\delta}\frac{1}{\rho u_\infty^2} = \frac{d}{dx}\left(\int_0^1\left(\frac{3}{2}\eta - \frac{\eta^3}{2}\right)\left[1-\left(\frac{3}{2}\eta - \frac{\eta^3}{2}\right)\right]\delta d\eta\right)$$

$$\Rightarrow \frac{3\nu}{2u_\infty\delta} = \frac{d\delta}{dx}\left(\frac{39}{280}\right) \qquad \Rightarrow \delta\frac{d\delta}{dx} = \left(\frac{280}{39}\right)\frac{3\nu}{2u_\infty}$$

Integral Method-IX

$$\Rightarrow \delta\frac{d\delta}{dx} = \left(\frac{280}{39}\right)\frac{3\nu}{2u_\infty}$$

Integration leads to $\frac{\delta^2}{2} = \frac{140\nu}{13u_\infty}x + C$

Using the condition $\delta = 0$ at $x = 0 \rightarrow C = 0$

$$\Rightarrow \delta^2 = \frac{280\nu}{13u_\infty}x \quad \Rightarrow \delta = 4.64\sqrt{\frac{\nu x}{u_\infty}} \quad \Rightarrow \frac{\delta}{x} = 4.64\sqrt{\frac{\nu}{u_\infty x}} \quad \Rightarrow \frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}}$$

Integral Method-X

$$\Rightarrow \frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}}$$

exact solution is $\frac{\delta}{x} = \frac{5.00}{\sqrt{Re_x}}$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{3\mu u_\infty/(2\delta)}{0.5\rho u_\infty^2} = \frac{3\mu}{\rho u_\infty \delta} = \frac{3\mu}{\rho u_\infty x}\frac{x}{\delta} = \frac{3}{Re_x}\frac{\sqrt{Re_x}}{4.64}$$

$$\Rightarrow C_f = \frac{0.646}{\sqrt{Re_x}}$$

The exact solution for this case is $C_f = \frac{0.664}{\sqrt{Re_x}}$

Integral Method-XII

- The integral Energy equation was derived as

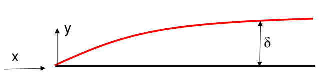
$$\Rightarrow \frac{d}{dx}\int_0^{\delta_T} u(T - T_\infty)dy = \frac{q''_w}{\rho c_p}$$

- The procedure is similar
- Since thermal boundary layer is smaller than velocity boundary layer, the velocity profile for this is same as previously derived
- The temperature profile is assumed as

$$T = a + by + cy^2 + dy^3$$

- The constants are evaluated using boundary conditions

Integral Method-XIII



$T = T_W$ at $y = 0$

$T = T_\infty$ at $y = \delta_T$

$\frac{dT}{dy} = 0$ at $y = \delta_T$

$\frac{d^2T}{dy^2} = 0$ at $y = 0$

- Evaluation of the constants lead to

$$\frac{T - T_\infty}{T_W - T_\infty} = 1 - \frac{3}{2}\frac{y}{\delta_T} + \frac{y^3}{2\delta_T^3}$$

- Further

$$q''_W = -k\left(\frac{dT}{dy}\right)_{y=0} = -k(T_W - T_\infty)\left(-\frac{3}{2\delta_T} + \frac{3y^2}{2\delta_T^3}\right)_{y=0} = \frac{k(T_W - T_\infty)}{\delta_T}\frac{3}{2}$$

Integral Method-XIV

- To simplify the algebra let us define $\eta_T = \frac{y}{\delta_T} \Rightarrow \delta_T d\eta_T = dy$

$$\frac{T - T_\infty}{T_W - T_\infty} = 1 - \frac{3}{2}\eta_T + \frac{\eta_T^3}{2}$$

Velocity Profile $\frac{u}{u_\infty} = \frac{3}{2}\eta - \frac{\eta^3}{2}$

$$\therefore \frac{q''_W}{\rho c_p} = \frac{3}{2}\frac{k(T_W - T_\infty)}{\delta_T \rho c_p} = u_\infty(T_W - T_\infty)\frac{d}{dx}\left(\int_0^{\delta_T}\frac{u}{u_\infty}\left(\frac{T - T_\infty}{T_W - T_\infty}\right)dy\right)$$

$$\Rightarrow \frac{3\alpha}{2\delta_T u_\infty} = \frac{d}{dx}\left(\int_0^1\left(\frac{3}{2}\eta - \frac{\eta^3}{2}\right)\left(1 - \frac{3}{2}\eta_T + \frac{\eta_T^3}{2}\right)\delta_T d\eta\right)$$

$$= \frac{d}{dx}\left(\int_0^1\left(\frac{3}{2}\eta_T\phi - \frac{\eta_T^3\phi^3}{2}\right)\left(1 - \frac{3}{2}\eta_T + \frac{\eta_T^3}{2}\right)\delta_T d\eta\right)$$

$\eta = \eta_T\phi$

$\phi = \frac{\delta_T}{\delta}$

Integral Method-XIV

$$\frac{3\alpha}{2\delta_T u_\infty} = \frac{d\delta_T}{dx}\left(\frac{3}{20}\phi - \frac{3\phi^3}{280}\right)$$

$$\delta_T \frac{d\delta_T}{dx} = \frac{3\alpha}{2u_\infty\left(\frac{3}{20}\phi - \frac{3\phi^3}{280}\right)}$$

Integration leads to $\frac{\delta_T^2}{2} = \frac{3\alpha}{2u_\infty\left(\frac{3}{20}\phi - \frac{3}{280}\phi^3\right)}x + C$

Using the condition $\delta = 0$ at $x = 0 \rightarrow C = 0$

Integral Method-XV

$$\delta_T = \sqrt{\frac{3\alpha x}{u_\infty\left(\frac{3}{20}\phi - \frac{3}{280}\phi^3\right)}}$$

We had shown that $\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}} \Rightarrow \delta = \frac{4.64}{\sqrt{u_\infty x/\nu}}x = \frac{4.64\sqrt{\nu x}}{\sqrt{u_\infty}}$

$$\Rightarrow \frac{\delta_T}{\delta} = \sqrt{\frac{3\alpha x}{u_\infty\left(\frac{3}{20}\phi - \frac{3}{280}\phi^3\right)}}\frac{\sqrt{u_\infty}}{4.64\sqrt{\nu x}} = \phi$$

$$\Rightarrow \phi^2 = \frac{3}{\left(\frac{3}{20}\phi - \frac{3}{280}\phi^3\right)4.64^2 \, Pr} \qquad \Rightarrow \phi^3 = \frac{0.923}{\left(1 - \frac{1}{14}\phi^2\right)Pr}$$

~0

Integral Method-XVI

$$\Rightarrow \therefore \phi = \frac{\delta_T}{\delta} = \frac{0.976}{Pr^{1/3}}$$

exact solution is $\frac{\delta_T}{\delta} = Pr^{-\frac{1}{3}}$

$$q'' = \frac{3}{2}k\frac{(T_W - T_\infty)}{\delta_T} = \frac{3}{2}k\frac{(T_W - T_\infty)}{\delta}\frac{\delta}{\delta_T}$$

$$\frac{q''}{(T_W - T_\infty)} = h = \frac{3}{2}k\frac{1}{\frac{4.64x}{\sqrt{Re_x}}}Pr^{1/3} = 0.3312\frac{k}{x}\sqrt{Re_x}\,Pr^{1/3}$$

$$\Rightarrow Nu_x = \frac{hx}{k} = 0.3312\sqrt{Re_x}\,Pr^{1/3}$$

Exact solution

Valid for 0.6 < Pr < 50 $Nu_x = 0.332\sqrt{Re_x}\,Pr^{1/3}$

Overview of Convection

- The study till now leads to the following summary
  - C_f is a function of Re
  - $Nu = f(Re, Pr)$ $Nu = C \; Re^m Pr^n$
  - $Sh = f(Re, Sc)$ $Sh = C \; Re^m Sc^n$
  - $Nu / Sh = Le^{-n}$
- In the integral method, we had seen that

$$C_f = \frac{0.646}{\sqrt{Re_x}} \qquad Nu_x = 0.3312\sqrt{Re_x}\,Pr^{1/3}$$

$$\Rightarrow \frac{C_f}{Nu_x} = \frac{0.646/\sqrt{Re_x}}{0.3312\sqrt{Re_x}\,Pr^{1/3}} \qquad \Rightarrow \frac{C_f}{2} \approx \frac{Nu_x}{Re_x\,Pr^{1/3}}$$

Modified Reynolds Analogy

Turbulent Convection-I

- Turbulence sets in at $Re_x = 5 \times 10^5$
- The relations without proof are given as follows

$$C_f(x) = 0.0592\,Re_x^{-0.2}$$

$$Nu_x = 0.0296\,Re_x^{0.8}\,Pr^{1/3}$$

- Note that Modified Reynolds analogy still holds
- The average values of friction coefficient and Nusselt number can be obtained from

$$\overline{C_f} = \frac{1}{L}\left[\int_0^{x_c} C_{fx\_lam}dx + \int_{x_c}^{L} C_{fx\_tur}dx\right] \qquad \overline{h} = \frac{1}{L}\left[\int_0^{x_c} h_{x\_lam}dx + \int_{x_c}^{L} h_{x\_tur}dx\right]$$

Turbulent Convection-II

- Substituting the expressions we can obtain the average values as

$$\overline{C_{f,L}} = \frac{0.074}{Re_L^{0.2}} - \frac{1742}{Re_L}$$

Assumes $Re_c = 5 \times 10^5$

$$\overline{Nu_L} = \left(0.037\,Re_L^{0.8} - 871\right)Pr^{1/3}$$

- If we assume that the fluid is turbulent right from the beginning, it can be shown that

$$\overline{C_{f,L}} = \frac{0.074}{Re_L^{0.2}}$$

$$\overline{Nu_L} = 0.037\,Re_L^{0.8}\,Pr^{1/3}$$

Other Cases of Heat Mass and Momentum Transfer

- We have seen the variation of C_f, Nu, Sh for the case of a flat plate
- These can be derived for cases such as over a cylinder, sphere, over series of cylinders, over series of spheres, etc.
- Relationships are similar, though it may be more complex
- Once the relationships are known, applications are straight forward
- Modified Reynolds analogy is invoked in many applications routinely