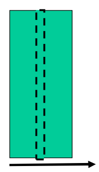
One Dimensional Steady Solutions-I

- As a first class of problem, we now turn to one dimensional analysis
- These are useful for large slabs, where transverse dimensions are much larger than thickness
- In such cases, Temperature is only a function of x.
- This implies q" can also be only a function of x.
- From thermodynamics, at steady state $q''_{in} = q''_{out}$ for any control volume
- This is equivalent to saying that heat flux is constant across any plane



One Dimensional Steady Solutions-II

- This is equivalent to saying

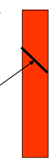
$$-k\frac{dT}{dx} = \text{constant}$$

- The same can be very easily obtained from heat equation

$$\Rightarrow \frac{\partial(\rho c T)}{\partial t} = \left(\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right) + q'''$$

$$\Rightarrow \frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \qquad \Rightarrow \left(k\frac{dT}{dx}\right) = C_1 \qquad \Rightarrow T = C_1'x + C_2$$

- The temperature distribution is linear

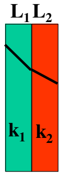


One Dimensional Steady Solutions-III

- Logically, if we had two slabs
  - The profile will be a piecewise straight line
  - In this case too, the heat flux along any plane will be constant

$$q'' = k_1\frac{T_1 - T_i}{L_1} = k_2\frac{T_i - T_2}{L_2}$$

  - Thus, knowing T_1, T_2, k_1, k_2, L_1, and L_2, T_i can be found and hence the temperature profile
  - To simplify analysis of these problems, we can introduce a principle similar to ohm's law



One Dimensional Steady Solutions-IV

- Heat flow is analogous to current, as it is same in series of slabs
- Temperature is analogous to voltage as it drives the current

$$i = q''A = k_1 A\frac{T_1 - T_i}{L_1} = k_2 A\frac{T_i - T_2}{L_2}$$

$$i = q = \frac{T_1 - T_i}{L_1/k_1 A} = \frac{T_i - T_2}{L_2/k_2 A} \qquad \Rightarrow q\left(L_1/k_1 A + L_2/k_2 A\right) = T_1 - T_2$$

- Thus Resistance is equivalent to L/(kA)
- q can be computed once, T_1, T_2, k_1, k_2, L_1, L_2, and A are known. T_i can then be computed once q is known

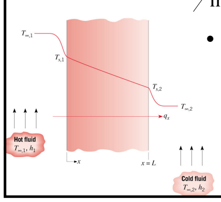
Convective Resistance

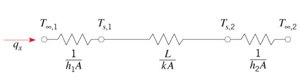
- If we also now consider convective boundary on either sides

$$q = h_1 A(T_{\infty 1} - T_{S1}) = \frac{(T_{S1} - T_{S2})}{L/kA} = h_2 A(T_{S2} - T_{\infty 2})$$

$$q = \frac{(T_{\infty 1} - T_{S1})}{1/h_1 A} = \frac{(T_{S1} - T_{S2})}{L/kA} = \frac{(T_{S2} - T_{\infty 2})}{1/h_2 A}$$

- Thus convective resistance is 1/(hA)





$$R_{eq} = 1/h_1 A + L/kA + 1/h_2 A$$

Radiation Resistance

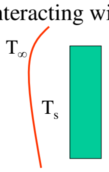
- If we had a radiation boundary on one side interacting with ambient at T_∞

$$q = \sigma\varepsilon A\left(T_\infty^4 - T_S^4\right)$$

$$\Rightarrow q = \sigma\varepsilon\left(T_\infty^2 + T_S^2\right)\left(T_\infty + T_S\right)A\left(T_\infty - T_S\right)$$

$$\Rightarrow q = h_r A\left(T_\infty - T_S\right)$$

$$\text{where } h_r = \sigma\varepsilon\left(T_\infty^2 + T_S^2\right)\left(T_\infty + T_S\right)$$

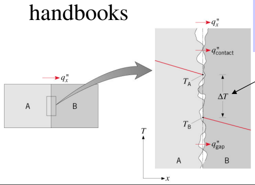


- Thus we can say radiation resistance can be written as $1/(h_r A)$
- In a restrictive case of convection and radiation boundary with the same ambient temperature, the boundary resistance shall be $1/((h_r+h_c)A)$

Contact Resistance

- When we had put slabs together, the interface is assumed to have one temperature
- In reality, the interface is complex and the heat transfer mode is complex too
- It can be idealized to be have a contact resistance with a finite jump in temperature as shown
- This is obtained experimentally and is tabulated in handbooks

$$qR_{t,c} = \Delta T_{contact} = \frac{q}{A} AR_{t,c} = q'' R''_{t,c}$$



- Sometimes the handbooks list contact conductance

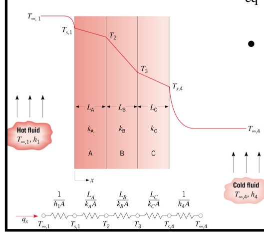
$$q = h_{contact} A \Delta T_{contact}$$

Overall Heat Transfer Coefficient

- When we have a series of slabs (contact resistance neglected)

$$q = \frac{(T_{\infty 1} - T_{\infty 2})}{R_{eq}}$$

$$R_{eq} = \frac{1}{h_1 A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{L_3}{k_3 A} + \frac{1}{h_2 A}$$

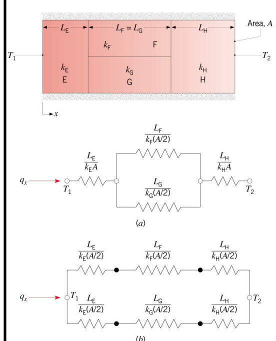


- In the heat transfer literature overall heat transfer coefficient, U, is defined as

$$q = UA(T_{\infty 1} - T_{\infty 2})$$

$$\Rightarrow UA = \frac{1}{R_{eq}}$$

Network Analysis



- Although one dimensional analysis is very restrictive, sometimes it is extended to a network
- This methodology is fast becoming obsolete due to the progress in numerical heat transfer

One Dimensional Cylindrical Wall

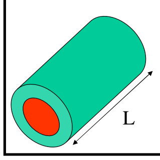
- The heat equation in cylindrical coordinate for a constant property system can be written as

$$\rho c \frac{\partial (T)}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right) + q'''$$

- In steady state with no heat generation and if the system can be assumed such that T varies only along r we can write

$$\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = 0 \qquad \Rightarrow rk\frac{dT}{dr} = C$$

$$\Rightarrow rq''_r = C \qquad \text{or} \Rightarrow 2\pi rLq''_r = C^* = q_r$$



This implies that the total heat crossing any radial plane in the cylinder is constant

Electrical Analogy-I

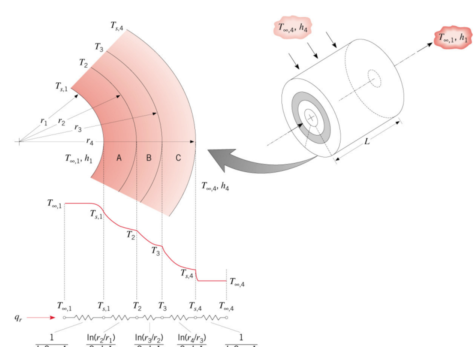
$$\Rightarrow 2\pi rL(-k)\frac{dT}{dr} = q_r \qquad \Rightarrow -dT = \frac{q_r}{2\pi rLk}dr$$

Integrating between R_i and R_o, we have

$$\int_{T_i}^{T_o} -dT = \int_{R_i}^{R_o} \frac{q_r}{2\pi rLk}dr \quad \Rightarrow T_i - T_o = \frac{q_r}{2\pi Lk}\ln\frac{R_o}{R_i}$$

Thermal resistance $\Rightarrow R_{cond} = \dfrac{\ln\dfrac{R_o}{R_i}}{2\pi Lk}$

Electrical Analogy-II



2

Computation of Temperature

- To compute temperature at any radius r within a cylindrical wall with T_o, T_i, R_o and R_i, we proceed as follows

$$T_i - T_o = \frac{q_r}{\dfrac{2\pi L k}{\ln\frac{R_o}{R_i}}} \quad \Rightarrow T_i - T_o = \frac{2\pi r L(-k)\frac{dT}{dr}}{\dfrac{2\pi L k}{\ln\frac{R_o}{R_i}}} \quad \Rightarrow T_i - T_o = -r\frac{dT}{dr}\ln\frac{R_o}{R_i}$$

- Now we integrate between R_i and R as follows:

$$\Rightarrow \frac{T_i - T_o}{\ln\frac{R_o}{R_i}}\int_{R_i}^{R}\frac{dr}{r} = -\int_{T_i}^{T} dT \qquad \Rightarrow \frac{T_i - T}{T_i - T_o} = \frac{\ln\frac{R}{R_i}}{\ln\frac{R_o}{R_i}}$$

One dimensional Spherical wall

- The heat equation in spherical coordinate for a constant property system can be written as

$$\rho c\frac{\partial (T)}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + q'''$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) = 0 \qquad \Rightarrow r^2 k\frac{\partial T}{\partial r} = C \qquad \Rightarrow r^2 q_r'' = C$$

$$\Rightarrow 4\pi r^2 q_r'' = C^* = q_r$$

This implies that the total heat crossing any radial plane in the sphere is constant

Electrical Analogy

$$4\pi r^2 q_r'' = q_r \qquad \Rightarrow 4\pi r^2\left(-k\frac{dT}{dr}\right) = q_r \qquad \Rightarrow -dT = \frac{q_r}{4\pi k}\frac{dr}{r^2}$$

Integrating between R_i and R_o, we have

$$\Rightarrow -\int_{T_i}^{T_o} dT = \frac{q_r}{4\pi k}\int_{R_i}^{R_o}\frac{dr}{r^2} \qquad \Rightarrow -T\Big|_{T_i}^{T_o} = \frac{q_r}{4\pi k}\frac{-1}{r}\Big|_{R_i}^{R_o}$$

$$\Rightarrow T_i - T_o = q_r\frac{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}{4\pi k}$$

Thermal resistance

Similarly, temperature at any R can be written as

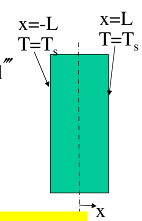
$$\Rightarrow \frac{T - T_i}{T_o - T_i} = \frac{\left(\frac{1}{r} - \frac{1}{r_i}\right)}{\left(\frac{1}{r_o} - \frac{1}{r_i}\right)}$$

Conduction with Heat Generation

- We had analyzed one-dimensional cases in slab, cylinder and sphere cases without heat generation
- In nuclear and other applications conduction in heat generating body is of important significance
- The heat generation can be uniform (electrical heating) or non-uniform (Radiation shields)
- The current crossing a plane is no longer a constant
- The heat generation affects the temperature distribution

1-D Slab with Heat Generation-I

$$\frac{\partial(\rho c T)}{\partial t} = \left(\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right) + q'''$$



$$\Rightarrow \frac{d}{dx}\left(k\frac{dT}{dx}\right) + q''' = 0 \qquad \text{For } -L\le x\le L$$

Boundary Conditions; At x=-L, T=T_s; At x=L, T=T_s

$$\Rightarrow \frac{d}{dx}\left(k\frac{dT}{dx}\right) = -q'''$$

We shall first restrict to constant volumetric heat generation rate

Integration of the above equation leads to

$$\Rightarrow \left(k\frac{dT}{dx}\right) = -q'''x + C_1$$

1-D Slab with Heat Generation-II

If k can be taken as constant, then $\Rightarrow \frac{dT}{dx} = \frac{-q'''}{k}x + C_1^*$

Integrating once more $\Rightarrow T = \frac{-q'''}{2k}x^2 + C_1^*x + C_2$

Boundary conditions imply

At x = -L $\quad T_s = \frac{-q'''}{2k}L^2 + C_1^*(-L) + C_2$

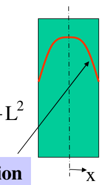
At x = L $\quad \Rightarrow T_s = \frac{-q'''}{2k}L^2 + C_1^*L + C_2$

Subtracting the above equations leads to C_1^*=0

Adding and simplifying leads to $\quad C_2 = T_s + \frac{q'''}{2k}L^2$

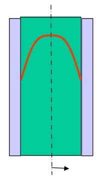
$$\Rightarrow T = \frac{q'''}{2k}\left(L^2 - x^2\right) + T_s$$

Parabolic distribution



1-D Slab with Heat Generation-III

- Now if we add a clad to it we can just modify the results in the following manner
- Now the surface of the clad temperature will be the boundary condition for the composite geometry
- Since no heat is generated in the clad, the heat rate crossing every plane will be constant and we can use the previous concepts derived
- The total heat rate crossing any clad plane is equal to heat rate generated in one half of the fuel



$$q = q'''AL$$

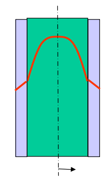
1-D Slab with Heat Generation-IV

- The temperature drop in clad is equal to

$$\Delta T_{clad} = q'''AL\frac{t}{k_c A} = \frac{q'''Lt}{k_c} \qquad \Rightarrow T_{ci} = T_{co} + \frac{q'''Lt}{k_c}$$

- The temperature anywhere inside the clad can be computed using linear interpolation as temperature profile is linear in the clad
- If we neglect the contact resistance, then T_s of fuel is same as that of T_{ci}.
- The fuel temperature can be now estimated from the equation derived previously

$$\Rightarrow T = \frac{q'''}{2k_f}\left(L^2 - x^2\right) + T_s$$



1-D Slab with Heat Generation-V

- The centerline temperature T_o can be computed as

$$\Rightarrow T_o = T_{co} + \frac{q'''Lt}{k_c} + \frac{q'''L^2}{2k_f}$$

- We can manipulate the above to get some interesting results

$$\Rightarrow T_o - T_{co} = q'''LA\left(\frac{t}{k_c A} + \frac{L}{2Ak_f}\right)$$

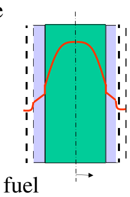
$\Delta T_{overall}$ q R_{clad} R_{fuel}

- This method is valid only to get T_o and valid only if k_f is constant

1-D Slab with Heat Generation-VI

- If we have clad, contact resistance and fluid cooling the fuel, then we can simply write

$$\Rightarrow T_o - T_\infty = q'''LA\left(\frac{1}{hA} + \frac{t}{k_c A} + \frac{R''_{contact}}{A} + \frac{L}{2Ak_f}\right)$$



- The Temperature profiles can be found in fuel once T_s is found, by using

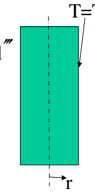
$$\Rightarrow T = \frac{q'''}{2k_f}\left(L^2 - x^2\right) + T_s$$

1-D cylinder with heat addition-I

- The analysis of cylinder is similar

$$\frac{\partial(\rho c T)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + q'''$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) + q''' = 0 \qquad \text{For } 0 \le r \le R$$



Boundary Conditions; At r=0, dT/dr=0; At r=R, T=T_s

$$\Rightarrow \frac{d}{dr}\left(rk\frac{dT}{dr}\right) = -q'''r$$

Restricting to constant volumetric heat generation rate

r=0
dT/dr=0

Integration of the above equation leads to

$$\Rightarrow rk\frac{dT}{dr} = -q'''\frac{r^2}{2} + C_1$$

1-D cylinder with heat addition-II

First Boundary condition implies $C_1 = 0$

$$\Rightarrow k\frac{dT}{dr} = -q'''\frac{r}{2} \qquad \Rightarrow \frac{dT}{dr} = -q'''\frac{r}{2k}$$

If k can be taken as constant, and we integrate once more

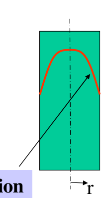
$$\Rightarrow T = \frac{-q'''}{4k}r^2 + C_2$$

Second Boundary condition implies

$$T_s = \frac{-q'''}{4k}R^2 + C_2$$

Or $C_2 = T_s + \frac{q'''}{4k}R^2$

$$\Rightarrow T = \frac{q'''}{4k}\left(R^2 - r^2\right) + T_s$$



$$\Rightarrow T_o = \frac{q'''R^2}{4k} + T_s$$

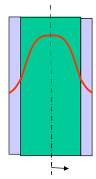
Parabolic distribution

1-D cylinder with heat addition-III

- Now if we add a clad to it we can just modify the results in the following manner
- The total heat rate crossing any clad radial plane is

$$q = q'''\pi R^2 L$$

- The temperature drop in clad is equal to



$$\Delta T_{clad} = q'''\pi R^2 L \frac{\ln\left(R_o / R_i\right)}{2\pi L k_c}$$

$$\Rightarrow T_{ci} = T_{co} + q''' R^2 \frac{\ln\left(R_o / R_i\right)}{2 k_c}$$

Clad temperature Profile

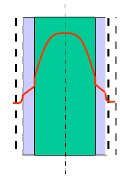
$$\frac{T_{ci} - T}{T_{ci} - T_{co}} = \frac{\ln \frac{R}{R_{ci}}}{\ln \frac{R_{co}}{R_{ci}}}$$

1-D cylinder with heat addition-IV

- If we have clad, contact resistance and fluid cooling the fuel, then we can simply write

$$\Rightarrow T_o - T_\infty = q'''\pi R_{ci}^2 L \left(\frac{1}{h 2\pi R_{co} L} + \frac{\ln\left(R_{co} / R_{ci}\right)}{2\pi L k_c} + \frac{R''_{contact}}{2\pi R_{ci} L} + \frac{1}{4\pi k_f L} \right)$$

Note that L, being arbitrary cancels out



1-D sphere with heat addition-I

- The analysis of sphere is similar

$$\rho c \cancel{\frac{\partial (T)}{\partial t}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(k r^2 \frac{\partial T}{\partial r}\right) + \cancel{\frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)} + \cancel{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k \sin\theta \frac{\partial T}{\partial \theta}\right)} + q'''$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 k \frac{dT}{dr}\right) + q''' = 0 \quad \text{For } 0 \le r \le R$$

Boundary Conditions; At r=0, dT/dr=0; At r=R, $T=T_s$

$$\Rightarrow \frac{d}{dr}\left(r^2 k \frac{dT}{dr}\right) = -q''' r^2$$

Restricting to constant volumetric heat generation rate

Integration of the above equation leads to

$$\Rightarrow r^2 k \frac{dT}{dr} = -q''' \frac{r^3}{3} + C_1$$

1-D sphere with heat addition-II

First Boundary condition implies $C_1 = 0$

$$\Rightarrow k\frac{dT}{dr} = -q''' \frac{r}{3} \qquad \Rightarrow \frac{dT}{dr} = -q''' \frac{r}{3k}$$

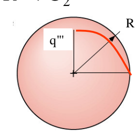
If k can be taken as constant, and we integrate once more

$$\Rightarrow T = \frac{-q'''}{6k} r^2 + C_2$$

Second Boundary condition implies

$$T_s = \frac{-q'''}{6k} R^2 + C_2$$

Or $C_2 = T_s + \frac{q'''}{6k} R^2$



$$\Rightarrow T = \frac{q'''}{6k}\left(R^2 - r^2\right) + T_s$$

$$\Rightarrow T_o = \frac{q''' R^2}{6k} + T_s$$

Parabolic distribution