Two-Dimensional Heat Transfer-I

- We Studied one dimensional heat transfer from plates, cylinders, spheres, Fins and Fin arrays
- Many real life applications are more complex and need 2-d or 3-d analysis.
- Three approaches are available, viz., analytical, graphical and numerical
- Analytical solution is exact, while graphical and numerical solutions are approximate
- Graphical solutions are obsolete and will not be discussed

Two-Dimensional Heat Transfer-II

- Analytical solution tends to become complex, but are used as benchmark solutions to qualify approximate numerical methods
- We shall restrict to two-dimensional cases in this course
- Complex three dimensional cases are now solved mostly with numerical codes
- We shall also restrict to Cartesian case. Complex cases can be studied in a course in Partial Differential Equations and Boundary Value Problems.

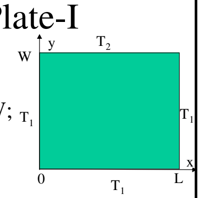
Two-Dimensional Heat Transfer-III

- The governing equation for 2-d steady heat transfer in Cartesian equations without heat generation is

$$\rho c\frac{\partial (T)}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+q''' \qquad \Rightarrow \frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

- The methodology that is used is called separation of variables
- This methodology is valid for linear equations with boundary conditions that can be separated
- Let us illustrate this for one problem

Heat Transfer in a Plate-I



- The governing equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \quad 0 \le x \le L;\ 0 \le y \le W;$$

- Boundary conditions

$$T(0,y) = T_1;\ T(L,y) = T_1;\ T(x,0) = T_1;\ T(x,W) = T_2$$

- To non-dimensionalize the temperature, we define

$$\theta = \frac{T - T_1}{T_2 - T_1} \quad \Rightarrow \frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=0 \quad 0 \le x \le L;\ 0 \le y \le W;$$

$$\theta(0,y) = 0;\ \theta(L,y) = 0;\ \theta(x,0) = 0;\ \theta(x,W) = 1$$

Heat Transfer in a Plate-II

- The solution for $\theta(x,y)$ is assumed of the form:

$$\theta(x,y) = X(x)Y(y)$$

- Substituting this in the governing equation, we get

$$\Rightarrow Y(y)\frac{d^2X(y)}{dx^2}+X(x)\frac{d^2Y(y)}{dy^2}=0 \qquad \Rightarrow -\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{Y}\frac{d^2Y}{dy^2}$$

- Since LHS is only a function of x and RHS is only a function of y and yet they be equal, would require that both sides be equal to a constant

$$\Rightarrow -\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{Y}\frac{d^2Y}{dy^2}=\lambda^2 \qquad \Rightarrow \frac{d^2X}{dx^2}+\lambda^2 X = 0; \quad \frac{d^2Y}{dy^2}-\lambda^2 Y = 0$$

Heat Transfer in a Plate-III

- The solution for X and Y are;

$$X = C_1\cos(\lambda x)+C_2\sin(\lambda x) \qquad Y = C_3 e^{-\lambda y}+C_4 e^{\lambda y}$$

$$\Rightarrow \theta = XY = \left(C_1\cos(\lambda x)+C_2\sin(\lambda x)\right)\left(C_3 e^{-\lambda y}+C_4 e^{\lambda y}\right)$$

- The BC at y =0; $\theta = 0$, at y = W; $\theta = 1$

  at x = 0; $\theta = 0$, at x = L; $\theta = 0$

BC at y =0 $\Rightarrow C_3 + C_4 = 0 \Rightarrow C_3 = -C_4$

BC at x =0 $\Rightarrow C_1 = 0$

$$\Rightarrow \theta = \left(C_2\sin(\lambda x)\right)C_4\left(e^{\lambda y}-e^{-\lambda y}\right)$$

We still have to satisfy two more conditions at x = L, y = W

BC at x = L $\Rightarrow \theta = 0 = \left(C_2\sin(\lambda L)\right)C_4\left(e^{\lambda y}-e^{-\lambda y}\right)$

Heat Transfer in a Plate-IV

- If either C_2 or C_4 is zero, then we get the trivial solution, which will violate the boundary condition at y = W, Hence not possible

$$\Rightarrow 0 = \sin(\lambda L) \Rightarrow \lambda L = n\pi; \quad n = 1,2,3,...\infty$$

$$\Rightarrow \lambda = \frac{n\pi}{L}; \quad n = 1,2,3,...\infty$$

$$\Rightarrow \theta = \left(C_2 \sin\left(\frac{n\pi x}{L}\right)\right) C_4 \left(e^{\left(\frac{n\pi x}{L}\right)y} - e^{-\left(\frac{n\pi x}{L}\right)y}\right); \quad n = 1,2,3,...\infty$$

$$\Rightarrow \theta = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

Heat Transfer in a Plate-V

- We still have to satisfy the boundary condition;

  At y = W; $\theta = 1$

- This involves concepts in orthogonal functions
- If we have several functions $f_1(x)$, $f_2(x)$, …$f_n(x)$, They are said to be orthogonal in a domain $a \le x \le b$, if

$$\int_a^b f_n(x) f_m(x)\,dx = 0 \qquad \text{for} \quad n \ne m$$

- It turns out that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0 \qquad \text{for} \quad n \ne m$$

Heat Transfer in a Plate-VI

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = L/2$$

- Further, $\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = \left(\frac{L}{n\pi}(1 - \cos(n\pi)\right)$
- We shall make use of all these to satisfy the BC at y = W

$$\Rightarrow \theta = 1 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi W}{L}\right)$$

- To make the above true for $0 \le x \le L$, we need to do some tricks to find the correct form of C_n. For this, the above expression is integrated over the domain to give

$$\int_0^L 1 \sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \sum_{n=1}^{\infty} C_n \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi W}{L}\right)dx$$

Heat Transfer in a Plate-VII

$$\int_0^L 1 \sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi W}{L}\right)\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

- Note that in the RHS, only term n=m will survive

$$\therefore \int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = C_n \sinh\left(\frac{n\pi W}{L}\right)\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow \frac{L}{n\pi}(1 - \cos n\pi) = C_n \sinh\left(\frac{n\pi W}{L}\right)\frac{L}{2}$$

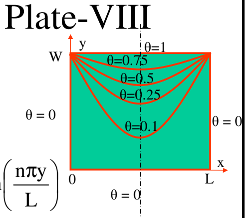
$$\Rightarrow C_n = \frac{2(1 - \cos(n\pi))}{n\pi \sinh\left(\frac{n\pi W}{L}\right)} = \frac{2(1 + (-1)^{n+1})}{n\pi \sinh\left(\frac{n\pi W}{L}\right)}$$

Heat Transfer in a Plate-VIII

- Hence the solution is

$$\Rightarrow \theta = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

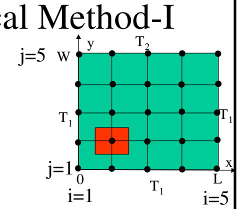
$$\Rightarrow \theta = \sum_{n=1}^{\infty} \frac{2(1 + (-1)^{n+1})}{n\pi \sinh\left(\frac{n\pi W}{L}\right)} \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$



- It may be observed that the solutions require fairly complex techniques. These become more complex in cylindrical and spherical coordinate systems. Compilation of these solutions are available in Hear transfer handbooks and a treatise by Carslaw and Jaeger

Elements of Numerical Method-I

- Numerical methods employ what is known as discrete mathematics
- In this method, the domain is discretized into discrete points and the solutions are found at these discrete locations



- The nodal temperatures are expected to be representative value for the control volume shown in red
- We represent any nodal temperature as T(i,j), where i increases along x direction and j increases along y direction

Elements of Numerical Method-II

- If we apply Taylor series, we can write

$$T(i+1,j) = T(i,j) + \left.\frac{\partial T}{\partial x}\right|_{i,j}\Delta x + \left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j}\frac{\Delta x^2}{2!} + \text{hot} \quad (1)$$

$$T(i-1,j) = T(i,j) - \left.\frac{\partial T}{\partial x}\right|_{i,j}\Delta x + \left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j}\frac{\Delta x^2}{2!} + \text{hot} \quad (2)$$

$$T(i,j+1) = T(i,j) + \left.\frac{\partial T}{\partial y}\right|_{i,j}\Delta y + \left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j}\frac{\Delta y^2}{2!} + \text{hot} \quad (3)$$

$$T(i,j-1) = T(i,j) - \left.\frac{\partial T}{\partial y}\right|_{i,j}\Delta y + \left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j}\frac{\Delta y^2}{2!} + \text{hot} \quad (4)$$

- Subtracting Eqs. (1) and (2) we get

$$T(i+1,j) - T(i-1,j) = 2\left.\frac{\partial T}{\partial x}\right|_{i,j}\Delta x + \text{hot} \quad \Rightarrow \frac{T(i+1,j) - T(i-1,j)}{2\Delta x} \approx \left.\frac{\partial T}{\partial x}\right|_{i,j} \quad (5)$$

Elements of Numerical Method-III

- Similarly Eqs. (3) and (4) we get

$$\Rightarrow \frac{T(i,j+1) - T(i,j-1)}{2\Delta y} \approx \left.\frac{\partial T}{\partial y}\right|_{i,j} \quad (6)$$

- Now adding Eqs (1) and (2) we get

$$T(i-1,j) + T(i+1,j) = 2T(i,j) + 2\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j}\frac{\Delta x^2}{2!} + \text{hot}$$

$$\Rightarrow \frac{T(i+1,j) - 2T(i,j) + T(i-1,j)}{\Delta x^2} \approx \left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \quad (7)$$

- Similarly adding Eqs (3) and (4) and simplifying. We get

$$\Rightarrow \frac{T(i,j+1) - 2T(i,j) + T(i,j-1)}{\Delta y^2} \approx \left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} \quad (8)$$

Elements of Numerical Method-IV

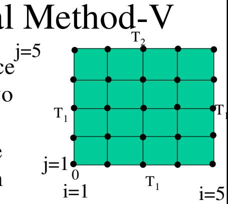
- In Eqs (5)-(8), we have obtained approximate expressions for the derivatives in terms of discrete values of temperature in the neighborhood
- These expressions are termed finite difference approximations of the derivatives
- The governing equation we want to solve is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

- Expressing the derivatives as finite differences for any arbitrary point i,j, we get

$$\frac{T(i+1,j) - 2T(i,j) + T(i-1,j)}{\Delta x^2} + \frac{T(i,j+1) - 2T(i,j) + T(i,j-1)}{\Delta y^2} = 0 \quad (9)$$

Elements of Numerical Method-V



- Eq. (9) is called the finite difference form of the Laplace equation in two dimensions
- Effectively, we have converted the partial differential equation into an algebraic equation
- For simplicity, if we choose Δx= Δy, then

$$T(i+1,j) - 2T(i,j) + T(i-1,j) + T(i,j+1) - 2T(i,j) + T(i,j-1) = 0$$

$$\Rightarrow T(i+1,j) + T(i-1,j) + T(i,j+1) + T(i,j-1) - 4T(i,j) = 0 \quad (10)$$

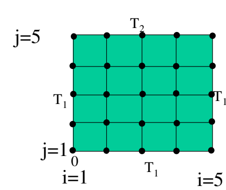
- For the given discretization as shown, we have 25 points, of which we know the 16 values as boundary conditions. Hence only 9 interior values need be found

Elements of Numerical Method-VI

- We can write a difference equation for each of the interior node and get 9 equations
- For the known boundary conditions, the 9 unknowns can be found by solving the set of 9 algebraic equations
- There are many techniques to solve these
- We shall illustrate the simplest of all called point Gauss-Siedel procedure
- Eq. (10) can be re-written as

$$T(i,j) = \frac{T(i+1,j) + T(i-1,j) + T(i,j+1) + T(i,j-1)}{4}$$

Elements of Numerical Method-VII



- The unknowns are for i = 2-4 and j = 2,4
- Assume arbitrary values for all unknowns, T(i,j) say T_1
- The algorithm will be

For i = 2, 4
For j = 2, 4

$$T(i,j) = \frac{T(i+1,j) + T(i-1,j) + T(i,j+1) + T(i,j-1)}{4}$$

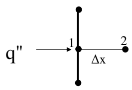
End j
End i

- Repeat till convergence $(T_{new} - T_{old})/T_{new} < \text{Tolerance}$

Elements of Numerical Method-VIII

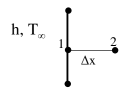
- When the boundary conditions are not the temperature specified ones, we need to write a finite difference equation for the boundary nodes
- Treatment of heat flux boundary condition

$$q'' = -k\frac{(T_2 - T_1)}{\Delta x} \quad \Rightarrow \frac{-q''\Delta x}{k} = T_2 - T_1$$



- Treatment of heat transfer boundary condition

$$h(T_\infty - T_1) = -k\frac{(T_2 - T_1)}{\Delta x} \quad \Rightarrow T_\infty = T_1\left(1 + \frac{k}{h\Delta x}\right) - T_2\frac{k}{h\Delta x}$$



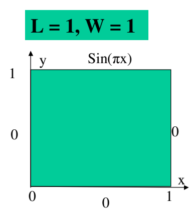
Elements of Numerical Method-IX

- Before closure, it is pertinent to point tat to obtain accurate solutions the number of nodes need to be increased
- Usual practice is to first get a coarse node solution with say 10 x 10 nodes. Then the process is repeated with 20 x 20 nodes. If the results are not much different, then no more refinement is done. Otherwise, the number of nodes is further increased till the results no longer change

An Example-I

- To illustrate the effectiveness of the numerical algorithm, let us take a very special boundary condition
- From our previous lecture on separation of variable, application of the 0 temperature boundary condition gave the solution as

$$\theta = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{1}\right)\sinh\left(\frac{n\pi y}{1}\right)$$



- The boundary condition at top wall (y = 1) gives

$$\sin(\pi x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\sinh(n\pi)$$

An Example-II

$$\sin(\pi x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\sinh(n\pi)$$

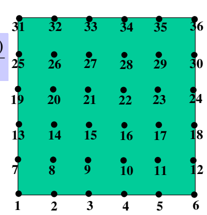
- Careful observation reveals that with n = 1, $C_1 = 1/\sinh(\pi)$ and all other $C_n = 0$, will exactly satisfy the boundary condition
- Therefore we can write the solution as

$$\theta = \frac{\sin(\pi x)\sinh(\pi y)}{\sinh(\pi)}$$

- It is very easy to verify that the above satisfies the differential equation and the boundary conditions

An Example-III

$$T(i,j) = \frac{T(i+1,j) + T(i-1,j) + T(i,j+1) + T(i,j-1)}{4}$$



- Started with arbitrary values for the interior as 0.5
- Performed 24 iterations
- Compared with analytical solution
- Maximum error less than 1.3%, when normalized with the maximum temperature