ME 704

Computational Methods in Thermal and Fluids Engineering

(Ordinary Differential Equations-I)

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Motivation

- Many of the physical laws leads to the development differential equations
  - Newton's Second Law : The acceleration produced is equal to Force/unit mass $d^2x/dt^2 = F/m$
  - First law of thermodynamics : The rate of change Energy of a system is equal to difference of the rate of heat and work transferred from the system $mc_v dT/dt = -h\ A\ (T-T_{amb})$

Some of the well known ODE's

- Dynamics

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f(t)$$

- Radioactivity

$$\frac{dN}{dt}=-\lambda N$$

- Conduction

$$\frac{d^2T}{dx^2}=-\frac{q'''}{k}$$

Classification of ODEs

- Order

$$\frac{dN}{dt}=-\lambda N$$ First Order

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f(t)$$ Second Order

- Linearity

$y'+y = 0$ Linear

$y'+y^2 = 0$ Non-linear

Classification of ODEs (cont'd)

Homogenity

$y' + y = 0$ Homogeneous

$y' + y = f(x)$ Non-Homogeneous

System of linear ODE's

$y' = f(x,y,z)$; $z' = g(x,y,z)$

- Analytical Solutions can be found for linear ODE's
- For most non-linear systems, numerical solution has to be resorted to

Classification of ODEs (cont'd)

Initial Value Problems (IVP)

The boundary conditions are specified at the same boundary

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

$$\text{with} \quad x(t=0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(t=0) = \dot{x}_0$$

Classification of ODEs (cont'd)

Boundary Value Problems (BVP)

The boundary conditions are specified at different boundaries

$$\frac{d^2T}{dx^2} = -\frac{q'''}{k}$$

$$\text{with} \quad T(x=0) = T_0 \quad \text{and} \quad T(x=L) = T_L$$

- The techniques for solution vary, though in principle, both problems can be solved by any one of the techniques

Solution of IVP for a first order ODE

- $y' = f(x,y)$, with $y(x=0) = y_0$
- Taylor Series Method

$$\bar{y}(x_0 + h) = \bar{y}(x_0) + \bar{y}'(x_0)h + \bar{y}''(x_0)\tfrac{h^2}{2!} + \ldots$$

$$+ \bar{y}^n(x_0)\tfrac{h^n}{n!} + \bar{y}^{n+1}(\xi)\tfrac{h^{n+1}}{n+1!}$$

$$\bar{y}(x_0) = y_0$$

$$\bar{y}'(x_0) = f(x_0, y_0)$$

$$\bar{y}'' = (\bar{y}')' = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\Rightarrow \bar{y}''(x_0) = f_x(x_0, \bar{y}_0) + f_y(x_0, \bar{y}_0) f(x_0, \bar{y}_0)$$

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Taylor Series Method (Cont'd)

- The method needs evaluation of derivatives
- Automation is cumbersome
- Estimation of error is difficult, if not impossible
- Not a preferred method

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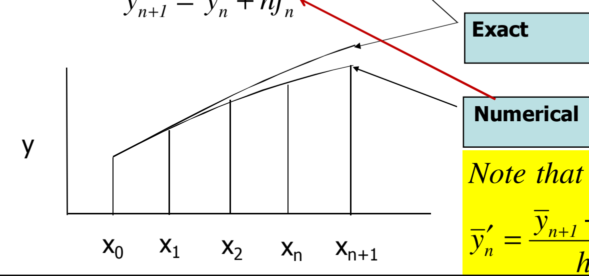
Euler's Method

This is the simplest method, wherein only the first two terms of the Taylor series is accounted for.

$$\bar{y}(x_0+h)=\bar{y}(x_0)+\bar{y}'(x_0)h+\bar{y}''(\xi)\tfrac{h^2}{2!}$$

$$\Rightarrow \begin{array}{l} \bar{y}_{n+1}=\bar{y}_n+hf_n+o(h^2) \\ y_{n+1}=y_n+hf_n \end{array}$$



Note that

$$\bar{y}_n'=\frac{\bar{y}_{n+1}-\bar{y}_n}{h}+o(h)$$

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Euler's Method (Cont'd)

When applied repetitively, it leads to

$$\bar{y}_N=y_0+\sum_{n=1}^{N}y_n-y_{n-1}+\sum_{n=1}^{N}y''(\xi)\tfrac{h^2}{2!} \qquad where \quad N=\frac{x_N-x_0}{h}$$

Error term

$$\sum_{n=1}^{N}y''(\xi)\tfrac{h^2}{2!}=Ny''(\xi)\tfrac{h^2}{2!}=\frac{x_N-x_0}{h}y''(\xi)\tfrac{h^2}{2!}$$

$$\Rightarrow \bar{y}_N=y_0+\sum_{n=1}^{N}y_n-y_{n-1}+o(h)$$

Globally First Order method

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Stability of Euler's Method

- Stability implies that the round off error should not explode
- This implies that the error be bounded, that is to say $e_{N+1}/\ e_N < 1$

$$\bar{y}_{N+1}=\bar{y}_N+hf(x_n,\bar{y}_n)+T_n \qquad \text{The exact value}$$

$$y_{N+1}=y_N+hf(x_n,y_n) \qquad \text{Numerical Estimate}$$

$$\Rightarrow \bar{y}_{N+1}-y_{N+1}=\bar{y}_N-y_N+h(f(x_n,\bar{y}_n)-f(x_n,y_n))+T_n$$

$$e_{N+1}=e_N+h\frac{(f(x_n,\bar{y}_n)-f(x_n,y_n))}{\bar{y}_N-y_N}(\bar{y}_N-y_N)$$

Note that the truncation error has been removed

Stability (cont'd)

In the limit h tending to zero

$$e_{N+1} = e_N + h\left(\frac{\partial f}{\partial y}\right)_N e_N$$

$$\Rightarrow \frac{e_{N+1}}{e_N} = \left[1 + h\left(\frac{\partial f}{\partial y}\right)_N\right]$$

The conditions for stability can be derived as

$$\left|\frac{e_{N+1}}{e_N}\right| = \left|1 + h\left(\frac{\partial f}{\partial y}\right)_N\right| \le 1$$

Stability (cont'd)

The previous condition leads to

$$-1 \le 1 + h\left(\frac{\partial f}{\partial y}\right)_N \le 1$$

$$1 + h\left(\frac{\partial f}{\partial y}\right)_N \le 1 \;\; \Rightarrow \frac{\partial f}{\partial y} \le 0$$

$$-1 \le 1 + h\left(\frac{\partial f}{\partial y}\right)_N \;\; \Rightarrow h \le \frac{2}{\left|\frac{\partial f}{\partial y}\right|} \;\; (note \frac{\partial f}{\partial y} \le 0)$$

Modified Euler's Method-I

- Higher order approximations can similarly be obtained.

$$\bar{y}_{n+1} = \bar{y}_n + \bar{y}'_n h + \bar{y}''_n \tfrac{h^2}{2!} + O(h^3)$$

$$\bar{y}''_n = \frac{\bar{y}'_{n+1} - \bar{y}'_n}{h} + O(h)$$

$$\Rightarrow y_{n+1} = \bar{y}_n + \bar{y}'_n h + \frac{h^2}{2}\frac{(\bar{y}'_{n+1} - \bar{y}'_n)}{h} + O(h^3)$$

$$\Rightarrow \bar{y}_{n+1} = \bar{y}_n + \frac{h}{2}(\bar{y}_{n+1} + \bar{y}'_n) + O(h^3)$$

$$\bar{y}_{n+1} = f(x, \bar{y})_{n+1}$$

- Thus y_{n+1} has to be estimated
- This is done by Euler's Method

Modified Euler's Method-II

- The overall method consists of the following two steps

$$y^p{}_{n+1} = y_n + hf(x_n, y_n)$$

$$y^c{}_{n+1} = y_n + h\frac{\left(f(x_n, y_n) + f(x_{n+1}, y^p_{n+1})\right)}{2}$$

Modified Euler's Method(Cont'd)

- It is a predictor-corrector method
- It requires two function evaluation per step
- The method is globally second order method
- Acceptable for some problems
- Not a preferred method
- Note that the slope used is the average estimated from point n and n+1
- This may be viewed as the slope computed as a weighted mean, with sum of the weights equal to one