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ME 704

Computational Methods in Thermal and Fluids Engineering

Solution of Elliptic Equations

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Common Elliptic Equations-I

- One of the most common elliptic equation is the Steady Heat Equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=-\frac{q'''}{k}$$

- Fully developed 2-D incompressible flow in ducts can be reduced to an elliptic equation as follows

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \Rightarrow v=0 \text{ everywhere}$$

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \qquad \Rightarrow \frac{\partial p}{\partial x}=\mu\frac{\partial^2 u}{\partial y^2}$$

$$\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \qquad \Rightarrow \frac{\partial p}{\partial y}=0 \Rightarrow p=p(x)$$

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Common Elliptic Equations-II

$$\underbrace{\frac{\partial p}{\partial x}}_{f(x)}=\mu\underbrace{\frac{\partial^2 u}{\partial y^2}}_{f(y)} \qquad \Rightarrow \frac{\partial^2 u}{\partial y^2}=\frac{1}{\mu}\frac{\partial p}{\partial x}=C$$

- A similar extension to 3-D with v=0 and w=0 leads to

$$\frac{\partial p}{\partial x}=\mu\left[\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial w^2}\right]$$

- In general, the above can be summed into

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0 \qquad \frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=f(x,y)$$

Laplace Eq. Poisson Eq.

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Common Elliptic Equations-III

- Comparing with our standard form

$$Af_{xx}+Bf_{xy}+Cf_{yy}+Df_x+Ef_y+F=0$$

$$B=0$$

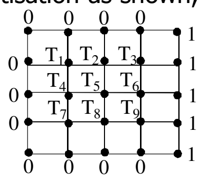
- For positive A and C, $B^2-4AC < 0$
- The equations are elliptic
  - No characteristic directions
  - No discontinuities
  - Information spreads in all directions
  - Every point affects every other point

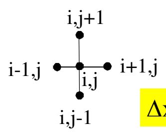
Solution of Elliptic Equations-I

- Consider Poisson Equation

$$\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] = S$$

- For a discretisation as shown,





Δx/Δy = β

- We can write it in finite difference form as

$$\frac{T^n_{i+1,j} - 2T^n_{i,j} + T^n_{i-1,j}}{\Delta x^2} + \frac{T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j-1}}{\Delta y^2} = S_{i,j}$$

Solution of Elliptic Equations-II

- The Finite Difference Equation can be written as

$$T_{i+1,j} - 2T_{i,j} + T_{i-1,j} + \left(T_{i,j+1} - 2T_{i,j} + T_{i,j-1}\right)\beta^2 = S_{i,j}\Delta x^2$$

- The same can be rearranged as

$$T_{i+1,j} + T_{i\ \ 1,j} + T_{i,j+1}\beta^2 + T_{i,j-1}\beta^2 - 2(1+\beta^2)T_{i,j} = S_{i,j}\Delta x^2$$

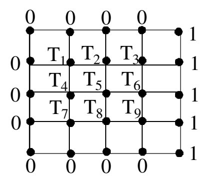
- For the simple case of S = 0, β =1, we get

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

- We can write the finite difference equation for each interior nodes and these can be solved as discussed

Solution of Elliptic Equations-III

- For the case shown in the grid, the relations can be described by the following Matrix problem



$$\begin{bmatrix} -4 & 1 & & 1 & & & & & \\ 1 & -4 & 1 & & 1 & & & & \\ & 1 & -4 & 1 & & 1 & & & \\ 1 & & 1 & -4 & 1 & & 1 & & \\ & 1 & & 1 & -4 & 1 & & 1 & \\ & & 1 & & 1 & -4 & 1 & & 1 \\ & & & 1 & & 1 & -4 & 1 & \\ & & & & 1 & & 1 & -4 & 1 \\ & & & & & 1 & & 1 & -4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

Banded Matrix

Solution of Elliptic Equations-IV

- Direct solution is not generally used.
- Several Iterative methods are common. These are:
  - Gauss-Siedel, Jacobi
  - Point Successive Over Relaxation (PSOR)
  - Line
  - Line-SOR
  - ADI
  - Accelerated ADI

Jacobi and Gauss Siedel

- Jacobi (for S = 0)

$$T_{i,j}^{k+1} = \frac{T_{i+1,j}^{k} + T_{i-1,j}^{k} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k}\right)\beta^2}{2\left(1+\beta^2\right)}$$

- Assume T^0 and iterate using the above relation
- If the new values available are immediately used for other points, it is called Gauss Siedel

$$T_{i,j}^{k+1} = \frac{T_{i+1,j}^{k} + T_{i-1,j}^{k+1} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1}\right)\beta^2}{2\left(1+\beta^2\right)}$$

- The necessary and sufficient condition for convergence is the Scarborough Criterion (Diagonal ≥ sum of off-diagonal and Diagonal > sum of off-diagonal at least in one row)

Line Method

- In point method, information flows from point to point
- Hence, the information from one boundary to another takes more time and it takes more effort to get convergence
- If we can facilitate exchange of information quickly, we can converge faster
- Line methods accomplish this effectively

$$T_{i-1,j}^{k+1} - 2\left(1+\beta^2\right)T_{i,j}^{k+1} + T_{i+1,j}^{k+1} = -\beta^2\left(T_{i,j-1}^{k+1} + T_{i,j+1}^{k}\right)$$

- The idea is to construct the Tri-diagonal matrix and solve them by TDMA

ADI Method

- In line by line the information from one set of boundaries moves fast but not from the other
- This can be accelerated, if we can alternate the direction
- This method is called Alternate Direction Implicit
- The iteration cycle has two parts.

X-Sweep

$$T_{i-1,j}^{k+0.5} - 2\left(1+\beta^2\right)T_{i,j}^{k+0.5} + T_{i+1,j}^{k+0.5} = -\beta^2\left(T_{i,j-1}^{k+0.5} + T_{i,j+1}^{k}\right)$$

Y-Sweep

$$\beta^2 T_{i,j-1}^{k+1} - 2\left(1+\beta^2\right)T_{i,j}^{k+1} + \beta^2 T_{i,j+1}^{k+1} = -T_{i+1,j}^{k+0.5} - T_{i-1,j}^{k+1}$$

Acceleration Procedures-I

- Point-SOR
  - The essence is that for a linear system it has been seen that we can accelerate by over relaxation

$$T_{i,j}^{k+1} = \alpha T_{i,j}^{k+1}\Big|_{GS} + (1-\alpha)T_{i,j}^{k}$$

  - The value of this over relaxation parameter, α, is > 1
  - Substituting and rearranging gives

$$T_{i,j}^{k+1} = T_{i,j}^{k} + \alpha\left(T_{i,j}^{k+1}\Big|_{GS} - T_{i,j}^{k}\right)$$

$$T_{i,j}^{k+1} = T_{i,j}^{k} + \frac{\alpha}{2\left(1+\beta^2\right)}\left(T_{i+1,j}^{k} + T_{i-1,j}^{k+1} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1}\right)\beta^2 - 2\left(1+\beta^2\right)T_{i,j}^{k}\right)$$

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Acceleration Procedures-II

- Line-SOR
  - The equation derived in the previous equation can be rearranged as

Same Eq.

$$T_{i,j}^{k+1} = T_{i,j}^{k} + \frac{\alpha}{2\left(1+\beta^2\right)}\left(T_{i+1,j}^{k} + T_{i-1,j}^{k+1} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1}\right)\beta^2 - 2\left(1+\beta^2\right)T_{i,j}^{k}\right)$$

$$2\left(1+\beta^2\right)T_{i,j}^{k+1} = 2\left(1+\beta^2\right)T_{i,j}^{k} + \alpha\left(T_{i+1,j}^{k} + T_{i-1,j}^{k+1} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1}\right)\beta^2 - 2\left(1+\beta^2\right)T_{i,j}^{k}\right)$$

$$\alpha T_{i+1,j}^{k+1} - 2\left(1+\beta^2\right)T_{i,j}^{k+1} + \alpha T_{i-1,j}^{k+1} = -(1-\alpha)2\left(1+\beta^2\right)T_{i,j}^{k} - \alpha\left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1}\right)\beta^2$$

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Acceleration Procedures-III

- Accelerated ADI
  - The equation derived for LSOR can be rearranged as

X-Sweep

$$\alpha T_{i+1,j}^{k+0.5} - 2\left(1+\beta^2\right)T_{i,j}^{k+0.5} + \alpha T_{i-1,j}^{k+0.5} = -(1-\alpha)2\left(1+\beta^2\right)T_{i,j}^{k} - \alpha\left(T_{i,j+1}^{k} + T_{i,j-1}^{k+0.5}\right)\beta^2$$

Y-Sweep

$$\alpha\beta^2 T_{i,j+1}^{k+1} - 2\left(1+\beta^2\right)T_{i,j}^{k+1} + \alpha\beta^2 T_{i,j-1}^{k+1} = -(1-\alpha)2\left(1+\beta^2\right)T_{i,j}^{k+0.5} - \alpha\left(T_{i+1,j}^{k+0.5} + T_{i-1,j}^{k+1}\right)$$

- Relative Comparison
  - For a 21X41 nodes for a slab of 1X2 with one of the smaller side held at 1, and for the same accuracy, results have been given in Hoffman and Chiang
  - It is reproduced in next slide

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Acceleration Procedures-IV

Relative Comparison

Method	Iterations	Time (CPU)	α_{opt}
GS (point)	574	5.524	
GS(Line)	308	7.196	
PSOR	52	1.082	1.78
LSOR	36	1.410	1.265
ADI	157	6.693	
AADI	23	1.535	1.27