MOC for a Set of Equations

- We have studied the method of characteristics for a single equation
- Now we shall extend this for a set of equations
- These find applications in Hydraulic Transients as applied to surge or water hammer
- While we shall see the introduction, more on it can be studied from Fluid transients by Wiley and Streeter
- We shall start from the governing equations we had derived in last lecture

The Governing Equations-I

- Conservation of Mass

$$\frac{\partial \rho A}{\partial t} + \frac{\partial \rho A V}{\partial x} = 0 \qquad (1)$$

- Conservation of Momentum

$$\frac{\partial \rho A V}{\partial t} + \frac{\partial \rho A V^2}{\partial x} = -A\frac{\partial p}{\partial x} - \tau_w P - \rho A g\frac{dH}{dx} \qquad (2)$$

A	Area
g	Gravitational acceleration
H	Elevation
P	Perimeter
p	Pressure

t	Time
V	Velocity
x	Spatial coordinate
ρ	Density of fluid
τ_w	Wall shear stress

The Governing Equations-II

$$A = \frac{\pi D^2}{4} \quad \Rightarrow dA = \frac{\pi\, 2D}{4}dD \quad \Rightarrow \frac{dA}{A} = 2\frac{dD}{D}$$

- Assuming thin cylinder theory, we can write hoops stress as

$$\sigma = \frac{pD}{2t} \quad \Rightarrow d\sigma \approx \frac{D}{2t}dp \Rightarrow E\ d\varepsilon \approx \frac{D}{2t}dp$$

$$\Rightarrow d\varepsilon \approx \frac{D}{2tE}dp \qquad d\varepsilon = \frac{dD}{D}$$

$$\Rightarrow \frac{dD}{D} = \frac{D}{2tE}dp = \frac{1}{2}\frac{dA}{A} \quad \Rightarrow \frac{dA}{dp} = \frac{AD}{tE} = \frac{\pi D^3}{4tE} \qquad (3)$$

- The above relation connects the change of area to a change in pressure

The Governing Equations-III

- In water hammer problems, the temperature of the water remains constant and hence the speed of sound can be defined as

$$\frac{\partial p}{\partial \rho} = \frac{dp}{d\rho} = a^2$$

- Now we can write

$$\frac{\partial(\rho A)}{\partial p} = \frac{d(\rho A)}{dp} = \rho\frac{dA}{dp} + A\frac{d\rho}{dp} = A\left(\rho\frac{1}{A}\frac{dA}{dp} + \frac{d\rho}{dp}\right)$$

$$\therefore \frac{d(\rho A)}{dp} = A\left(\rho\frac{1}{A}\frac{dA}{dp} + \frac{1}{a^2}\right) = \frac{A}{\hat{a}^2}$$

- The above relation involves the modified speed of sound due to elasticity of structure

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The Governing Equations-IV

- Now we are ready to manipulate the equations towards obtaining the solution

- Conservation of Mass can be modified as follows

$$\frac{\partial \rho A}{\partial t}+\frac{\partial \rho A V}{\partial x}=0 \qquad \Rightarrow \frac{d(\rho A)}{dp}\frac{\partial p}{\partial t}+V\frac{\partial \rho A}{\partial x}+\rho A\frac{\partial V}{\partial x}=0$$

$$\frac{d(\rho A)}{dp}\frac{\partial p}{\partial t}+V\frac{d(\rho A)}{dp}\frac{\partial p}{\partial x}+\rho A\frac{\partial V}{\partial x}=0$$

$$\frac{d(\rho A)}{dp}\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)+\rho A\frac{\partial V}{\partial x}=0$$

$$\frac{A}{\hat{a}^2}\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)+\rho A\frac{\partial V}{\partial x}=0 \Rightarrow \frac{1}{\hat{a}^2}\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)+\rho\frac{\partial V}{\partial x}=0$$

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The Governing Equations-V

- Conservation of Momentum Can be modified as

$$\frac{\partial \rho A V}{\partial t}+\frac{\partial \rho A V^2}{\partial x}=-A\frac{\partial p}{\partial x}-\tau_w P-\rho A g\frac{dH}{dx}$$

$$\rho A\frac{\partial V}{\partial t}+\rho A V\frac{\partial V}{\partial x}+\left(V\left[\frac{\partial \rho A}{\partial t}+V\frac{\partial \rho A V}{\partial x}\right]\right)=-A\frac{\partial p}{\partial x}-\tau_w P-\rho A g\frac{dH}{dx}$$

$$\rho A\frac{\partial V}{\partial t}+\rho A V\frac{\partial V}{\partial x}+A\frac{\partial p}{\partial x}=\frac{-1}{2}\rho V|V|fP-\rho A g\frac{dH}{dx}$$

- Dividing by ρA and putting P/A = 4/D, we get

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=-\frac{4V|V|f}{2D}-g\frac{dH}{dx}$$

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The Governing Equations-VI

- In our classification lecture, we had taken a special case where gravitational and frictional effects were ignored, the equation set was shown to be hyperbolic.
- The above two equations can be recast as

$$\begin{bmatrix}\frac{1}{\hat{a}^2} & 0\\ 0 & 1\end{bmatrix}\frac{\partial}{\partial t}\begin{Bmatrix}p\\V\end{Bmatrix}+\begin{bmatrix}\frac{V}{\hat{a}^2} & \rho\\ \frac{1}{\rho} & V\end{bmatrix}\frac{\partial}{\partial x}\begin{Bmatrix}p\\V\end{Bmatrix}=\begin{Bmatrix}\frac{-4fV|V|}{2D}-g\frac{dH}{dx}\end{Bmatrix}$$

- Using B-λA = 0, we can get

$$\Rightarrow \lambda = V \pm \hat{a} = \frac{dx}{dt}$$

- We will do the same by an alternate way

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Method of Lagrange Multiplier

- If L_1 = 0 and L_2 = 0 are the continuity and momentum equations, then we can also write

$$L_1+\beta\ L_2$$

$$\Rightarrow \frac{1}{\hat{a}^2}\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)+\rho\frac{\partial V}{\partial x}+$$

$$\beta\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4\rho V|V|f}{2D}+g\frac{dH}{dx}\right)=0$$

$$\Rightarrow \frac{1}{\hat{a}^2}\frac{\partial p}{\partial t}+\frac{\partial p}{\partial x}\left(\frac{V}{\hat{a}^2}+\frac{\beta}{\rho}\right)+\beta\frac{\partial V}{\partial t}+(\rho+\beta V)\frac{\partial V}{\partial x}+$$

$$\beta\left(\frac{4\rho V|V|f}{2D}+g\frac{dH}{dx}\right)=0$$

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Method of Lagrange Multiplier

$$\Rightarrow \frac{1}{\hat{a}^2}\left[\frac{\partial p}{\partial t}+\frac{\partial p}{\partial x}\left(\frac{V}{\hat{a}^2}+\frac{\beta}{\rho}\right)\hat{a}^2\right]+\beta\left[\frac{\partial V}{\partial t}+\frac{(\rho+\beta V)}{\beta}\frac{\partial V}{\partial x}\right]+$$
$$\beta\left(\frac{4\rho V|V|f}{2D}+g\frac{dH}{dx}\right)=0$$

- We can now state using the continuity concepts discussed earlier that we can write the terms in square parenthesis as total derivative along

$$\frac{dx}{dt}=\left(\frac{V}{\hat{a}^2}+\frac{\beta}{\rho}\right)\hat{a}^2=\frac{(\rho+\beta V)}{\beta}=\lambda \quad \textbf{(4)}$$

$$\Rightarrow V+\frac{\beta}{\rho}\hat{a}^2=\frac{\rho}{\beta}+V \quad \Rightarrow \beta^2=\frac{\rho^2}{\hat{a}^2} \quad \Rightarrow \beta=\pm\frac{\rho}{\hat{a}} \quad \textbf{(5)}$$

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Method of Lagrange Multiplier-I

- From Eqs. (4) and (5) we can write

$$\frac{dx}{dt}=\left(\frac{V}{\hat{a}^2}+\frac{\beta}{\rho}\right)\hat{a}^2=V\pm\hat{a}$$

- Now we can write the compatibility equations as

$$Along \quad \frac{dx}{dt}=V+\hat{a} \quad or \quad \beta=\frac{\rho}{\hat{a}}$$

$$\Rightarrow \frac{1}{\hat{a}^2}\frac{dp}{dt}+\beta\frac{dV}{dt}+\beta\left(\frac{4\rho V|V|f}{2D}+g\frac{dH}{dx}\right)=0$$

$$\Rightarrow \frac{1}{\hat{a}^2\beta}\frac{dp}{dt}+\frac{dV}{dt}=-\left(\frac{4\rho V|V|f}{2D}+g\frac{dH}{dx}\right)$$

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Method of Lagrange Multiplier-II

- Substituting for β

$$\Rightarrow \frac{1}{\rho\hat{a}}\frac{dp}{dt}+\frac{dV}{dt}=-\left(\frac{4\rho V|V|f}{2D}+g\frac{dH}{dx}\right)$$

- Similarly,

$$Along \quad \frac{dx}{dt}=V-\hat{a} \quad or \quad \beta=-\frac{\rho}{\hat{a}}$$

$$\Rightarrow -\frac{1}{\rho\hat{a}}\frac{dp}{dt}+\frac{dV}{dt}=-\left(\frac{4\rho V|V|f}{2D}+g\frac{dH}{dx}\right)$$

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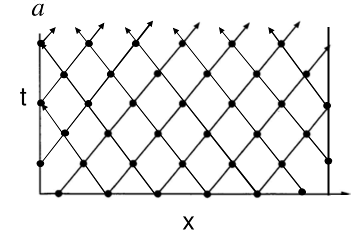
Characteristic Grid

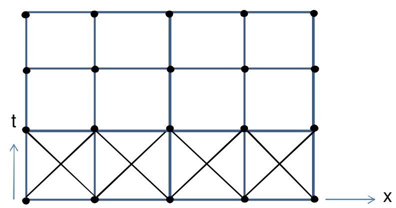
- From Every Point two characteristics emerge
- These represent the directions along which the signals propagate and also represent directions along which discontinuities might exist.
- In water hammer cases

$$V << \hat{a} \quad \Rightarrow \frac{dt}{dx}=\pm\frac{1}{\hat{a}}$$



Numerical MOC

- Since effective speed of sound is constant, programming is easy.
- We can choose Δt such that $\hat{a}\Delta t / \Delta x = 1$ This would imply that charcteristics from the previous time step exactly pass through the points in the next time step



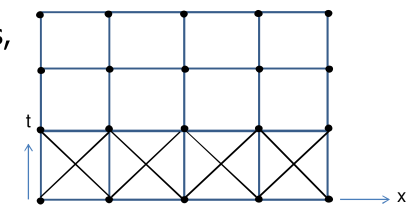
Numerical MOC

- Solution for interior nodes

$$Along \quad \frac{dt}{dx} = \frac{1}{\hat{a}} \quad \Rightarrow \frac{1}{\rho\hat{a}} \frac{p_i^{n+1} - p_{i-1}^{n}}{\Delta t} + \frac{V_i^{n+1} - V_{i-1}^{n}}{\Delta t} = RHS_{\,i-1}$$

$$Along \quad \frac{dt}{dx} = -\frac{1}{\hat{a}} \quad \Rightarrow -\frac{1}{\rho\hat{a}} \frac{p_i^{n+1} - p_{i+1}^{n}}{\Delta t} + \frac{V_i^{n+1} - V_{i+1}^{n}}{\Delta t} = RHS_{\,i+1}$$

- The two unknowns, viz., velocity and pressure at new time step can be found



Numerical MOC

- Treatment of Boundary conditions
- At boundary only one of the characteristic is relevant
- We can find either p or v depending on what is specified.