Final Set of Equations

Mass Balance

$$\frac{\partial(A\rho)}{\partial t}+\frac{\partial(A\rho V)}{\partial s}=0$$

Momentum Balance

$$\rho\frac{\partial(V)}{\partial t}+\rho V\frac{\partial(V)}{\partial s}=-\frac{\partial p}{\partial s}-\tau_w\frac{P}{A}-\rho g\frac{\partial H}{\partial s}$$

Energy Balance

$$\rho A\frac{\partial h}{\partial t}+\rho A u\frac{\partial h}{\partial s}=VA\frac{\partial p}{\partial s}+A\frac{\partial p}{\partial t}+q''_{surface}P-\frac{\partial(q''_{axial}A)}{\partial s}-W'$$

MOC for Compressible Flows

- We have studied the method of characteristics for water hammer equations
- In that case the density was treated as constant except for accounting of sonic speed
- However in compressible flows the assumption of constant density will not be valid. Since density is also dictated by temperature, energy equation has to be considered
- However, the pipes can be treated as rigid and hence modified speed of sound need not be accounted

The Governing Equations-I

- We will study for constant area duct
- The gravitational term can be neglected
- We will neglect axial conduction in energy Equation
  - Conservation of Mass

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho V}{\partial x}=0 \qquad (1)$$

  - Conservation of Momentum

$$\rho\frac{\partial V}{\partial t}+\rho V\frac{\partial V}{\partial x}+\frac{\partial p}{\partial x}=-\frac{\tau_w P}{A} \qquad (2)$$

The Governing Equations-II

- Conservation of energy can be written as

$$\rho\frac{\partial h}{\partial t}+\rho V\frac{\partial h}{\partial x}-\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)=\frac{\tau_w P}{A}V+\frac{q''P}{A} \qquad (3)$$

  - We have assumed an insulated system
  - Axial conduction neglected
  - No shaft work
- Expressing h = h(p,ρ), we can write

$$\Delta h=\left.\frac{\partial h}{\partial p}\right|_\rho\Delta p+\left.\frac{\partial h}{\partial \rho}\right|_p\Delta\rho \qquad \frac{\partial h}{\partial t}=\left.\frac{\partial h}{\partial p}\right|_\rho\frac{\partial p}{\partial t}+\left.\frac{\partial h}{\partial \rho}\right|_p\frac{\partial \rho}{\partial t}$$

2:34 PM 5/18

The Governing Equations-III

- Similarly, we can write

$$\frac{\partial h}{\partial x}=\left.\frac{\partial h}{\partial p}\right|_{\rho}\frac{\partial p}{\partial x}+\left.\frac{\partial h}{\partial \rho}\right|_{p}\frac{\partial \rho}{\partial x}$$

- To keep the notation simple, we shall define

$$\left.\frac{\partial h}{\partial p}\right|_{\rho}=h_p,\quad \left.\frac{\partial h}{\partial \rho}\right|_{p}=h_\rho$$

$$\Rightarrow\frac{\partial h}{\partial t}=h_p\frac{\partial p}{\partial t}+h_\rho\frac{\partial \rho}{\partial t}$$ and $$\Rightarrow\frac{\partial h}{\partial x}=h_p\frac{\partial p}{\partial x}+h_\rho\frac{\partial \rho}{\partial x}$$

2:34 PM 6/18

The Governing Equations-IV

- The first two terms of energy equation can now be written as

$$\rho\frac{\partial h}{\partial t}+\rho V\frac{\partial h}{\partial x}=\rho\left(h_\rho\frac{\partial \rho}{\partial t}+h_p\frac{\partial p}{\partial t}\right)+\rho V\left(h_\rho\frac{\partial \rho}{\partial x}+h_p\frac{\partial p}{\partial x}\right)$$

- Thus, the LHS of energy equation can now be written as

$$\rho\left(h_\rho\frac{\partial \rho}{\partial t}+h_p\frac{\partial p}{\partial t}\right)+\rho V\left(h_\rho\frac{\partial \rho}{\partial x}+h_p\frac{\partial p}{\partial x}\right)-\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)$$

- Collecting terms and equating to RHS, we get

$$\left(\rho h_p-1\right)\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)+\rho h_\rho\left(\frac{\partial \rho}{\partial t}+V\frac{\partial \rho}{\partial x}\right)=\frac{\tau_w P}{A}V+\frac{q'' P}{A}$$

2:34 PM 7/18

The Governing Equations-V

- Thus, the energy equation may be written as

$$\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)+\frac{\rho h_\rho}{\left(\rho h_p-1\right)}\left(\frac{\partial \rho}{\partial t}+V\frac{\partial \rho}{\partial x}\right)=\frac{1}{\left(\rho h_p-1\right)}\left(\frac{\tau_w P}{A}V+\frac{q'' P}{A}\right)$$

- It can be shown that

$$\frac{\rho h_\rho}{\left(\rho h_p-1\right)}=-a^2$$

- The final form of energy equation is

$$\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)-a^2\left(\frac{\partial \rho}{\partial t}+V\frac{\partial \rho}{\partial x}\right)=\frac{1}{\left(\rho h_p-1\right)}\left(\frac{\tau_w P}{A}V+\frac{q'' P}{A}\right) \quad (4)$$

2:34 PM 8/18

The Governing Equations-VI

- The energy equation is already in the characteristic form

$$\frac{dp}{dt}-a^2\frac{d\rho}{dt}=\frac{1}{\left(\rho h_p-1\right)}\left(\frac{\tau_w P}{A}V+\frac{q'' P}{A}\right)$$

along

$$\frac{dx}{dt}=V$$

- To get the continuity and momentum in the characteristic form, in pressure and velocity, we do some algebraic manipulations. Eq. (4) gives,

$$\left(\frac{\partial \rho}{\partial t}+V\frac{\partial \rho}{\partial x}\right)=\left[\left(\frac{\partial p}{\partial t}+V\frac{\partial p}{\partial x}\right)-RHS_E\right]\frac{1}{a^2} \quad (5)$$

The Governing Equations-VII

- Where RHS_E is given by

$$RHS_E = \frac{1}{(\rho h_p - 1)}\left(\frac{\tau_w P}{A} V + \frac{q'' P}{A}\right)$$

- Conservation of Mass (Eq. (1)) is rewritten as

$$\frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial x} + \rho\frac{\partial V}{\partial x} = 0 \quad (6)$$

- Eqs. (5) and (6) gives

$$\left[\left(\frac{\partial p}{\partial t} + V\frac{\partial p}{\partial x}\right) - RHS_E\right]\frac{1}{a^2} + \rho\frac{\partial V}{\partial x} = 0$$

$$\Rightarrow \left(\frac{\partial p}{\partial t} + V\frac{\partial p}{\partial x}\right) + a^2\rho\frac{\partial V}{\partial x} - RHS_E = 0 \quad (7)$$

MOC-I

- Conservation of Momentum Eq. (2) is rewritten as

$$\rho\frac{\partial V}{\partial t} + \rho V\frac{\partial V}{\partial x} + \frac{\partial p}{\partial x} + \frac{\tau_w P}{A} = 0 \quad (8)$$

- Using the Lagrange multiplier route as before Eq. (7)+β Eq. (8) = 0 implies

$$\left(\frac{\partial p}{\partial t} + V\frac{\partial p}{\partial x}\right) + a^2\rho\frac{\partial V}{\partial x} - RHS_E + \beta\left(\rho\frac{\partial V}{\partial t} + \rho V\frac{\partial V}{\partial x} + \frac{\partial p}{\partial x} + \frac{\tau_w P}{A}\right) = 0$$

$$\Rightarrow \frac{\partial p}{\partial t} + (\beta + V)\frac{\partial p}{\partial x} + \beta\rho\left(\frac{\partial V}{\partial t} + \left(V + \frac{a^2}{\beta}\right)\frac{\partial V}{\partial x}\right) - RHS_E - \beta\frac{\tau_w P}{A} = 0$$

MOC-II

- The characteristic directions shall be given by

$$\frac{dx}{dt} = V + \frac{a^2}{\beta} = \beta + V = \lambda \qquad \Rightarrow \beta = \pm a$$

$$\Rightarrow \frac{dx}{dt} = V \pm a$$

- The compatibility conditions are

$$\Rightarrow \frac{dp}{dt} \pm a\rho\frac{dV}{dt} = RHS_E \pm a\frac{\tau_w P}{A} = 0, \quad along\ \frac{dx}{dt} = V \pm a$$

MOC-III

- Before proceeding further, let us summarise the modified governing equations
- The combined mass and momentum equations can be written as

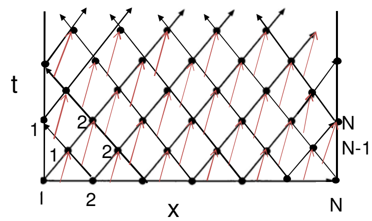
$$\frac{dp}{dt} \pm a\rho\frac{dV}{dt} = RHS_E \pm a\frac{\tau_w P}{A} = 0, \quad along\ \frac{dt}{dx} = \frac{1}{V \pm a}$$

- The energy equation is

$$\frac{dp}{dt} - a^2\frac{d\rho}{dt} = RHS_E, \quad along\ \frac{dt}{dx} = \frac{1}{V}$$

Forward Marching-I

- From every point three characteristics emerge
- The directions 1/(V+a), 1/(V-a) are called Mach lines and the direction 1/V is called the path line
- Just as in water hammer equations, the variation of pressure and velocity are computed from Mach lines
- The change in density is then computed from path lines
- Let us look at the forward method, which is rather complex



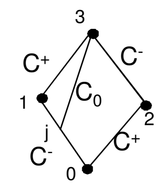
Forward Marching-II

- The values of x, t, p, V and ρ are assumed to be known at points 0, 1 and 2. Also, if p and ρ are known, so is ‘a’

$$\frac{t_3 - t_1}{x_3 - x_1} = \frac{1}{V_1 + a_1} \qquad \frac{t_3 - t_2}{x_3 - x_2} = \frac{1}{V_2 - a_2}$$

Get x_3 and t_3

- Using compatibility along 1-3 and 2-3, compute p_3 and V_3 similar to the method used in water hammer equations

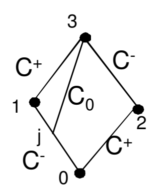


Forward Marching-III

- Using the following relations, we can determine x_j ,V_j and t_j

$$\frac{t_j - t_0}{x_j - x_0} = \frac{1}{V_0 - a_0} \qquad \frac{t_3 - t_j}{x_3 - x_j} = \frac{1}{V_j} \qquad \frac{x_j - x_0}{x_1 - x_0} = \frac{V_j - V_0}{V_1 - V_0}$$

- Using values of x_j and parameters at 0 and 1 compute, p_j and ρ_j by linear interpolation . Using the compatibility along j-3, and known p_3, p_j and ρ_j we can find ρ_3

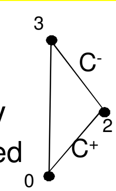


Forward Marching-IV

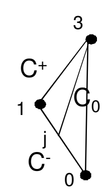
- Boundary conditions are determined similarly

Left Boundary

p, ρ known,V computed



Right Boundary

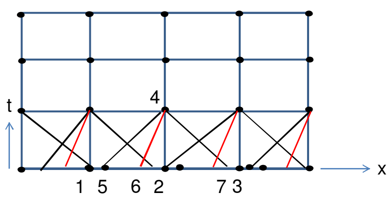


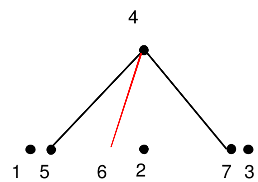
p known, V and ρ computed

- If the flow chokes at exit, Mach No must be set equal to one and pressure computed
- The logic is fairly tedious.

Backward Marching-IV

- This is designed for structured grid





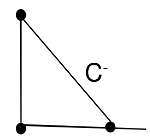
Steps

1. Compute 'a' at 1 and 3
2. Calculate coordinates of 5 and 7

$$\frac{t_4 - t_5}{x_4 - x_5} = \frac{1}{V_1 + a_1} \qquad \frac{t_4 - t_7}{x_4 - x_7} = \frac{1}{V_3 - a_3}$$

3. Interpolate all variables to get them at 5 and 7
4. Solve for p_4 and V_4
5. Calculate x coordinate at 6 using $\frac{t_4 - t_6}{x_4 - x_6} = \frac{1}{V_4}$
6. Calculate ρ_4
7. Boundary conditions are solved for similarly

Left Boundary



Right Boundary