ME-704 –CMTFE
Solution of Navier-Stokes Equations-I

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TRANSPORT EQUATIONS

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0$$

$$\frac{\partial \rho u}{\partial t}+\frac{\partial}{\partial x}\left\langle \rho uu - \mu\frac{\partial u}{\partial x}\right\rangle+\frac{\partial}{\partial y}\left\langle \rho uv - \mu\frac{\partial u}{\partial y}\right\rangle = \frac{\partial p}{\partial x}+\rho g_x$$

$$\frac{\partial \rho v}{\partial t}+\frac{\partial}{\partial x}\left\langle \rho uv - \mu\frac{\partial v}{\partial x}\right\rangle+\frac{\partial}{\partial y}\left\langle \rho vv - \mu\frac{\partial v}{\partial y}\right\rangle = \frac{\partial p}{\partial y}+\rho g_y$$

- In the above set p = p/ρ

Solution Methods-I

- The general methods available can be divided into three class of methods
  1. Method that eliminates pressure
  2. Method that derives a Poisson equation for pressure correction denoted as pressure based methods
  3. Method that modifies the continuity equation arbitrarily to create an equation for pressure. This method is called the Pseudo Compressibility method
- We will now start with the method that eliminates pressure and is called the stream function – vorticity method

Solution Methods-II

- This method is applicable only for 2-D flows and was the most popular method in 60s
- Even today, if we need only 2-D solutions this method is the best one as:
  1. It is the easiest to code
  2. Has the least numerical complications and behaves very well
  3. Probably the best for class room exercise
- Let us look at the details of this method
- We will stick only with incompressible flows

Solution Methods-IV

- The primitive form of the NS equations in the absence of body forces and with uniform viscosity can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

- To eliminate pressure, we cross differentiate the momentum equations and subtract one from the other

- The cross differentiation leads to

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right)$$

- To simplify matters, let us first look at LHS

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\overset{1}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}+\overset{2}{u\frac{\partial^2 v}{\partial x^2}}\right)-\left(\overset{3}{\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}}+\overset{4}{u\frac{\partial^2 u}{\partial x\partial y}}\right)+$$

$$\left(\underset{5}{\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}+\underset{6}{v\frac{\partial^2 v}{\partial x\partial y}}\right)-\left(\underset{7}{\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}}+\underset{8}{v\frac{\partial^2 u}{\partial x^2}}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\overset{1\ 3}{\cancel{\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}}+\overset{2\ 4}{u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}+$$

$$\underset{5\ 7}{\cancel{\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}}+\underset{6\ 8}{v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}$$

Using Continuity

- From fluid mechanics, we can write,

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

- Thus LHS reduces to $\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}$

- Looking at RHS

$$\frac{\partial}{\partial x}\left(-\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right)-\frac{\partial}{\partial y}\left(-\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right)$$

$$\Rightarrow\nu\left(\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right)$$

$$\Rightarrow\nu\left(\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right)$$

$$\Rightarrow\nu\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)$$

- Thus the two momentum equations collectively reduce to

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial x^2}\right)$$

- This is nothing but the transport of vorticity and is called the Vorticity Transport Equation
- Further we had shown previously that the stream function, ψ, satisfies the continuity automatically and vorticity can be rewritten as

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \Rightarrow \omega = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$\Rightarrow \omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

- Thus, the governing equations of fluid motion are:

$$\Rightarrow \omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial x^2}\right)$$

- The velocities can be expressed as

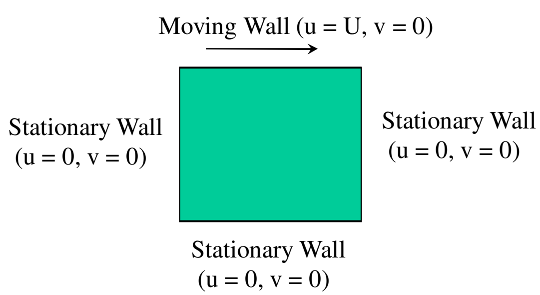
$$\Rightarrow u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

- Since we have already discussed methods to solve transport and Poisson equations, no new concept is required

Solution Methodology-I

Driven Cavity Problem

- This is a standard problem with which every Navier-Stokes Solvers are usually tested



Solution Methodology-II

Boundary Conditions for ψ

- Along vertical walls horizontal velocity is 0

$$\Rightarrow u = \frac{\partial \psi}{\partial y} = 0$$

- This implies that ψ is constant all along the horizontal wall
- Similarly, along horizontal Walls vertical velocity is 0

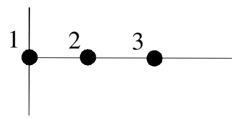
$$\Rightarrow v = -\frac{\partial \psi}{\partial x} = 0$$

- This implies that ψ is constant all along the vertical wall

Solution Methodology-III

Boundary Conditions for ω

- Let us consider the left wall (1,j)



- Since v = 0 on the wall

$$\Rightarrow \left.\frac{\partial \psi}{\partial x}\right|_{1,j} = 0$$

- Using Taylor Series, we can write

$$\Rightarrow \psi_{1,j} = \psi_{1,j} + \left.\frac{\partial \psi}{\partial x}\right|_{1,j} \Delta x + \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} \frac{\Delta x^2}{2}$$

Solution Methodology-IV

- Also, since u = 0 all along the wall $\Rightarrow \frac{\partial \psi}{\partial y} = 0$

$$\Rightarrow \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{along the left wall}$$

- By definition $\Rightarrow \omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$

$$\Rightarrow \omega_{1,j} = -\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j}$$

- From the above equation and the Taylor series expansion given in previous slide, we can write

$$\Rightarrow \psi_{2,j} = \psi_{1,j} + \left.\frac{\partial \psi}{\partial x}\right|_{1,j} \Delta x + \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} \frac{\Delta x^2}{2} = 0 + 0 - \omega_{1,j} \frac{\Delta x^2}{2}$$

- Thus, we can write at left boundary

$$\Rightarrow \omega_{1,j} = -\psi_{2,j} \frac{2}{\Delta x^2}$$

- Similarly at the right boundary

$$\Rightarrow \omega_{Nx,j} = -\psi_{Nx-1,j} \frac{2}{\Delta x^2}$$

- Similarly, we can write at bottom boundary

$$\Rightarrow \omega_{i,1} = -\psi_{i,2} \frac{2}{\Delta y^2}$$

- Similarly for the top boundary

$$\Rightarrow \omega_{i,Ny} = -\psi_{i,ny-1} \frac{2}{\Delta y^2} - \frac{2U}{\Delta y}$$

The difference is due to top u = U

Details of the Procedure

1. Specify geometry, initial conditions, top velocity and property values. You may note that the values of u, v, ψ, and ω are all 0 at the initial condition
2. FTCS or BTCS can be used for the transport equation, which must satisfy the stability condition
3. Solve the transport equation to get ω^{n+1}
4. Now that the ω has been found everywhere, solve for stream function equation to get ψ^{n+1} . You can use Central difference scheme as done previously for elliptical solver
5. Having found ψ, compute for u^{n+1} and v^{n+1} using a central difference representation for the derivative

Details of the Procedure

6. Obtain the boundary conditions for ω^{n+1}
7. You would have to iterate steps 3, 4, 5 and 6 in case of implicit methods being used.
8. March to the next time step and continue till the entire time of interest is completed, or steady state is reached.