ME-704 –CMTFE-KNI-15

Introduction to Control Volume Formulation

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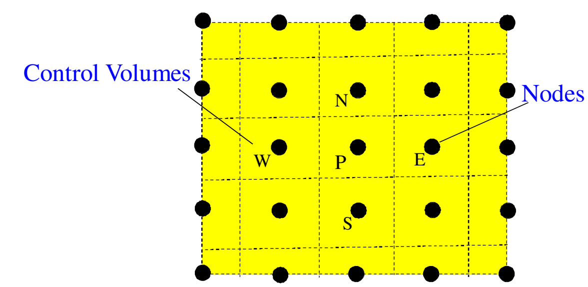
TRANSPORT EQUATIONS

Generalised Transport Equation

$$\frac{\partial(\rho\phi)}{\partial t}+\frac{\partial}{\partial x}\left(\rho u\phi-\Gamma\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\rho v\phi-\Gamma\frac{\partial\phi}{\partial y}\right)=S$$

Conservation Equation	Transported variable	Diffusion Coefficient	Source Term
Mass	1	0	0
X-momentum	u	μ	$-\frac{\partial p}{\partial x}+\rho g_x$
Y-momentum	v	μ	$-\frac{\partial p}{\partial y}+\rho g_y$

DOMAIN DISCRETISATION

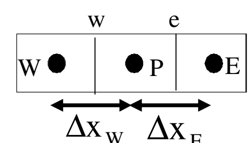


- The domain is divided into nodes and control faces are kept at the middle of these nodes
- The problem is to solve for the unknowns at the nodes

DISCRETISATION PRINCIPLES-1

Consider 1-D Convection-Diffusion Equation

$$\frac{d}{dx}\left(\rho u\phi-\Gamma\frac{\partial\phi}{\partial x}\right)=0$$



Control Volume Integration implies

$$\int_w^e \frac{d}{dx}\left(\rho u\phi-\Gamma\frac{d\phi}{dx}\right)dx=0$$

2:48 PM 5/28

DISCRETISATION PRINCIPLES-II

$$\left(\rho u\phi - \Gamma\frac{\partial\phi}{\partial x}\right)_e - \left(\rho u\phi - \Gamma\frac{\partial\phi}{\partial x}\right)_w = 0 \quad (1)$$

For linear variation of the variable, we can write

$$\phi_e = \frac{\phi_E + \phi_P}{2},\quad \phi_w = \frac{\phi_W + \phi_P}{2},\quad \left.\frac{\partial\phi}{\partial x}\right|_e = \frac{\phi_E - \phi_P}{\Delta x_E},\quad \left.\frac{\partial\phi}{\partial x}\right|_w = \frac{\phi_P - \phi_W}{\Delta x_W}$$

Substitution results in

$$\phi_E\left(\frac{\rho u|_e}{2} - \frac{\Gamma_e}{\Delta x_E}\right) - \phi_W\left(-\frac{\rho u|_w}{2} - \frac{\Gamma_w}{\Delta x_W}\right) + \phi_P\left(\frac{\rho u|_e}{2} + \frac{\Gamma_e}{\Delta x_E} - \frac{\rho u|_w}{2} + \frac{\Gamma_w}{\Delta x_W}\right) = 0$$

Central Difference Scheme

2:48 PM 6/28

DISCRETISATION PRINCIPLES-III

Difficulties of Central Difference

The previous equation can be written as

$$a_p\phi_P = a_E\phi_E + a_W\phi_W,$$

$$where,\quad a_E = \left(\frac{\Gamma_e}{\Delta x_E} - \frac{\rho u|_e}{2}\right),$$

$$a_W = \left(\frac{\Gamma_w}{\Delta x_W} + \frac{\rho u|_w}{2}\right)$$

$$a_P = \left(a_E + a_W + \rho u|_e - \rho u|_w\right)$$

2:48 PM 7/28

DISCRETISATION PRINCIPLES-IV

- For every node, a similar equation can be written
- If boundaries are known, we can write N equations for N interior nodes and the problem can be solved
- But the solution is found to be oscillatory for cases with large convection
- This was the overshoot problem

2:48 PM 8/28

REASONS FOR THE DIFFICULTY

The coefficient for a_E can be written as

$$a_E = \frac{\Gamma_e}{\Delta x_E}\left(1 - \frac{\rho u|_e \Delta x_E}{2\Gamma_e}\right) = \frac{\Gamma_e}{\Delta x_E}\left(1 - \frac{P_e}{2}\right)$$

where p_e is defined as the cell Peclet number

The second term in the bracket can become negative at large velocities.

A negative weight can give unphysical solutions

e.g. (50 X 2 + 100 X (-4))/(2-4) = -150

THE REMEDY

- Better schemes can be derived from the analytical solution of the convection-diffusion equation

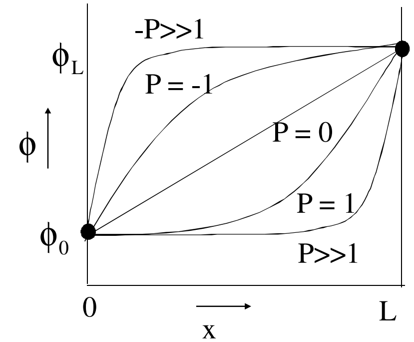
The analytical solution subject to boundary conditions,

$$\phi(x=0)=\phi_0,\quad \phi(x=L)=\phi_L$$

is given by

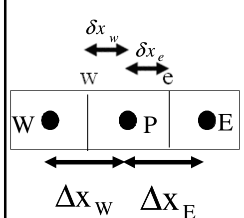
$$\frac{\phi-\phi_0}{\phi_L-\phi_0}=\left(\frac{e^{\frac{\rho u x}{\Gamma}}-1}{e^{\frac{\rho u L}{\Gamma}}-1}\right)$$

ANALYTICAL SOLUTION



EXPONENTIAL SCHEME-I

The values of the function and its derivative at the control surface 'e' can be obtained as



$$\phi_e=\phi_P+(\phi_E-\phi_P)\left(\frac{e^{\frac{\rho u|_e\,\delta x_e}{\Gamma_e}}-1}{e^{\frac{\rho u|_e\,\Delta x_E}{\Gamma_e}}-1}\right)$$

$$\left.\frac{\partial\phi}{\partial x}\right|_e=(\phi_E-\phi_P)\left(\frac{e^{\frac{\rho u|_e\,\delta x_e}{\Gamma_e}}}{e^{\frac{\rho u|_e\,\Delta x_E}{\Gamma_e}}-1}\right)\frac{\rho u|_e}{\Gamma_e}$$

EXPONENTIAL SCHEME-II

- Similarly the values at w can be computed and these can be substituted in Eq. (1) of slide 5 to get

$$a_p\phi_P=a_E\phi_E+a_W\phi_W,$$

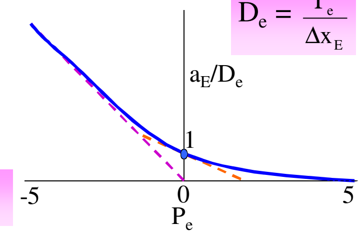
where, $a_E=\left(\frac{\rho u|_e}{e^{P_e}-1}\right)$, $a_W=\left(\frac{\rho u|_w\, e^{P_w}}{e^{P_w}-1}\right)$ and $a_P=\left(a_E+a_W+\rho u|_e-\rho u|_w\right)$

$$D_e=\frac{\Gamma_e}{\Delta x_E}$$

a_E can be rewritten as

$$a_E=\frac{\Gamma_e}{\Delta x_E}\left(\frac{P_e}{e^{P_e}-1}\right)$$

Note: a_E ➡ 0 as $P_e >> 1$



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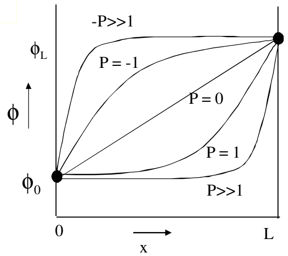
UPWIND SCHEME

When convection dominates

$\phi_e = \phi_P$ and $\phi_w = \phi_W$
for u >> 0

$\phi_e = \phi_E$ and $\phi_w = \phi_P$
for u << 0



- Evaluating gradient by linear variation, we can write,

$$a_p\phi_P = a_E\phi_E + a_W\phi_W,$$

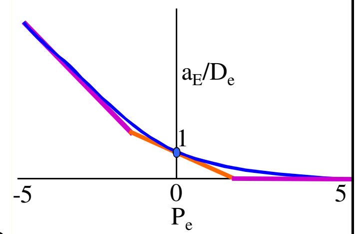
where, $a_E = D_e + \langle -\rho u|_e, 0\rangle$, $a_W = D_w + \langle \rho u|_w, 0\rangle$

and $a_P = (a_E + a_W + \rho u|_e - \rho u|_w)$,

2:48 PM 14/28

HYBRID SCHEME

- Properties of analytical solution of a_E/D_e are:
  - $a_E/D_e = 1$ at $P_e = 0$
  - $a_E/D_e \rightarrow 0$ as $P_e >> 1$
  - $a_E/D_e \rightarrow -P_e$ as $-P_e >> 1$
  - $\frac{d(a_E/D_e)}{dP_e} = -0.5$ at $P_e = 0$



$$a_p\phi_P = a_E\phi_E + a_W\phi_W,$$

where, $a_E = \left\langle -\rho u|_e, D_e - \frac{\rho u|_e}{2}, 0\right\rangle$, $a_W = \left\langle \rho u|_w, D_w + \frac{\rho u|_w}{2}, 0\right\rangle$

and $a_P = (a_E + a_W + \rho u|_e - \rho u|_w)$,

2:48 PM 15/28

GENERALISATION OF ALL SCHEMES

All the schemes can be generalised using a function $A(|P|)$

$$a_p\phi_P = a_E\phi_E + a_W\phi_W,$$

where, $a_E = D_e A(|P_e|) + \langle -F_e, 0\rangle$,

$a_W = D_w A(|P_w|) + \langle F_w, 0\rangle$ and

$a_P = (a_E + a_W + F_e - F_w)$,

Scheme	$A(\|P\|)$
Central	$1-0.5\ \|P\|$
Upwind	1
Hybrid	$\langle 0,\ 1-0.5\|P\|\rangle$
Exponential	$\frac{\|P\|}{e^{\|P\|}-1}$

2:48 PM 16/28

PRESSURE LINKED EQUATIONS

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}\left(\rho uu - \mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho uu - \mu\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}\left(\rho uv - \mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho uv - \mu\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \rho g_y$$

There is no transport equation for p

2:48 PM 17/28

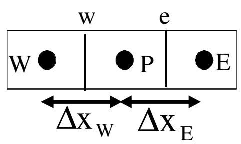
CHECKERBOARDING PROBLEM

- Early attempts at the solution led to oscillatory pressure fields
- A possible interpretation is given below

Control volume integration of pressure term leads to

$$\int_w^e -\frac{\partial p}{\partial x}dx = p|_w - p|_e$$

$p_e = 0.5(p_P + p_E)$ and $p_w = 0.5(p_W + p_P)$



$$\int_w^e -\frac{\partial p}{\partial x}dx = 0.5(p_W - p_E)$$

Equal increase of pressure in alternate nodes will not affect velocity

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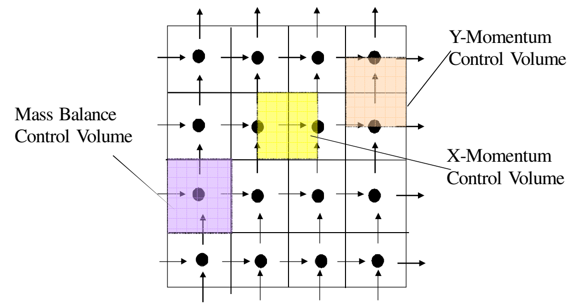
REMEDY – THE STAGGERED GRID



No interpolation of pressure required

2:48 PM 19/28

DRAWBACKS OF STAGGERED GRID

- Indices have to be carefully kept track
- Large number of coefficients have to be carried along

- Non-staggered Methods have been developed
- The principle is to manipulate the pressure gradient terms

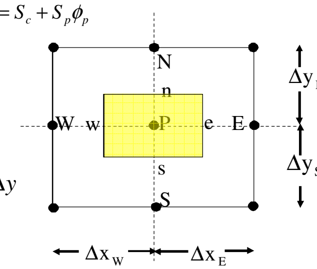
2:48 PM 20/28

Discretisation of 2-D Transport Equation

$$\frac{\partial \rho\phi}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\phi - \Gamma\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho u\phi - \Gamma\frac{\partial \phi}{\partial y}\right) = S_c + S_p\phi_p$$

$$\iint \frac{\partial \rho\phi}{\partial t}dxdy = \frac{\rho\phi|_P^{\Delta t} - \rho\phi|_P^0}{\Delta t}\Delta x\Delta y$$

$$\iint \left(S_c + S_p\phi_P\right)dxdy = \left[S_c + S_p\phi_P\right]\Delta x\Delta y$$



$$\iint \frac{\partial}{\partial y}\left(\rho u\phi - \Gamma\frac{\partial \phi}{\partial y}\right)dxdy = \left[\left(\rho u\phi - \Gamma\frac{\partial \phi}{\partial y}\right)_n - \left(\rho u\phi - \Gamma\frac{\partial \phi}{\partial x}\right)_s\right]\Delta x$$

$$\iint \frac{\partial}{\partial x}\left(\rho u\phi - \Gamma\frac{\partial \phi}{\partial x}\right)dxdy = \left[\left(\rho u\phi - \Gamma\frac{\partial \phi}{\partial x}\right)_e - \left(\rho u\phi - \Gamma\frac{\partial \phi}{\partial x}\right)_w\right]\Delta y$$

- Similar discretisation of mass balance will yield

$$\frac{\rho|_P^{\Delta t} - \rho|_P^{0}}{\Delta t}\Delta x\Delta y + \left(\rho u|_e - \rho u|_w\right)\Delta y + \left(\rho u|_n - \rho u|_s\right)\Delta x = 0$$

- Multiplying the above with $\varphi|_P^{\Delta t}$ and subtracting from the discretised momentum equation, we get

$$\frac{\phi|_P^{\Delta t} - \phi|_P^{0}}{\Delta t}\rho_p^0\Delta x\Delta y + \left[\left(\rho u\phi - \Gamma\frac{\partial\phi}{\partial x}\right)_e - \left(\rho u\phi - \Gamma\frac{\partial\phi}{\partial x}\right)_w - \left(\rho u|_e - \rho u|_w\right)\phi_P\right]\Delta y +$$
$$\left[\left(\rho u\phi - \Gamma\frac{\partial\phi}{\partial y}\right)_n - \left(\rho u\phi - \Gamma\frac{\partial\phi}{\partial x}\right)_s - \left(\rho u|_n - \rho u|_s\right)\phi_P\right]\Delta x = \left(S_c + S_p\phi_P\right)\Delta x\Delta y$$

- Plugging the profiles as done earlier, we get

2:48 PM 21

2:48 PM 22/28

Discretised of 2-D Transport Equation (Cont'd)

$$a_p\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + b$$

$$where,\ a_E = D_eA(|P_e|) + \langle -F_e, 0\rangle,\ a_W = D_wA(|P_w|) + \langle F_w, 0\rangle\ ,$$
$$a_N = D_nA(|P_n|) + \langle -F_n, 0\rangle,\ a_S = D_sA(|P_s|) + \langle F_s, 0\rangle\ ,$$
$$b = S_c\Delta x\Delta y + \frac{\rho_P^{\Delta t}\phi_p\Delta x\Delta y}{\Delta t}$$
$$and\quad a_P = \left(a_E + a_W + a_N + a_S + \frac{\rho_P^0\Delta x\Delta y}{\Delta t} - S_P\Delta x\Delta y\right),$$
$$D_e = \frac{\Gamma_e\Delta y}{\Delta x_E},\ D_w = \frac{\Gamma_w\Delta y}{\Delta x_W},\ D_n = \frac{\Gamma_n\Delta x}{\Delta y_N},\ D_s = \frac{\Gamma_s\Delta x}{\Delta y_S},$$
$$F_e = \rho u|_e\Delta y,\ F_w = \rho u|_w\Delta y,\ F_n = \rho u|_n\Delta x,\ F_s = \rho u|_s\Delta x,$$
$$P_e = \frac{F_e}{D_e},\ P_w = \frac{F_w}{D_w},\ P_n = \frac{F_n}{D_n},\ P_s = \frac{F_s}{D_s}.$$

2:48 PM 23/28

SIMPLE(Semi-IMplicit procedure for solving Pressure Linked Equations)

1. Assume a pressure field and velocity field
2. Compute coefficients for x and y momentum equations
3. Solve for u's and v's using x and y momentum equations
4. Check whether it satisfies mass balance
5. Obtain a new guess for pressure field systematically so that the resulting velocity would satisfy mass balance
6. Repeat steps 2-5 till convergence is obtained

2:48 PM 24/28

The Pressure Correction Equation

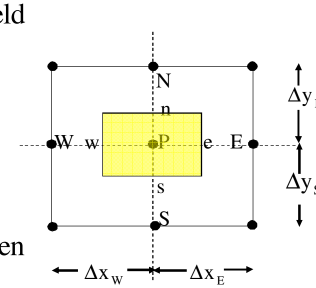
Let p^* be the assumed pressure field

X-momentum

$$a_e u_e^* = \sum a_{nb}u_{nb}^* + b_e + (p_P^* - p_E^*)\Delta y$$

Y-momentum

$$a_n v_n^* = \sum a_{nb}v_{nb}^* + b_n + (p_P^* - p_N^*)\Delta x$$



If p is the correct pressure field then

$$a_e u_e = \sum a_{nb}u_{nb} + b_e + (p_P - p_E)\Delta y$$

$$a_n v_n = \sum a_{nb}v_{nb} + b_n + (p_P - p_N)\Delta x$$

Subtraction of the approximate fields from the correct fields lead to

2:48 PM 25/28

The Pressure Correction Equation (cont'd)

$$a_e u'_e = \sum a_{nb} u'_{nb} + (p'_P - p'_E)\Delta y \qquad a_n v'_n = \sum a_{nb} v'_{nb} + (p'_P - p'_N)\Delta x$$

$$u'_e = (p'_P - p'_E)\frac{\Delta y}{a_e} \qquad v'_n = (p'_P - p'_N)\frac{\Delta x}{a_n}$$

Control volume integration of mass balance leads to

$$\frac{\rho|_P^{\Delta t} - \rho|_P^{0}}{\Delta t}\Delta x\Delta y + \left(\rho_e(u_e^* + u'_e) - \rho_w(u_w^* + u'_w)\right)\Delta y +$$
$$\left(\rho_e(v_n^* + v'_n) - \rho_s(v_s^* + v'_s)\right)\Delta x = 0$$

Substitution of u_e', v_n' from above and rearranging leads to

2:48 PM 26/28

The Pressure Correction Equation (cont'd)

$$a_p p'_P = a_E p'_P + a_W p'_W + a_N p'_N + a_S p'_S + b$$

where, $a_E = \rho_e \frac{\Delta y}{a_e}\Delta y,\ a_W = \rho_w \frac{\Delta y}{a_w}\Delta y,\ a_N = \rho_n \frac{\Delta x}{a_n}\Delta x,\ a_S = \rho_s \frac{\Delta x}{a_s}\Delta x,$

and $a_P = a_E + a_W + a_N + a_S,$

$$b = \left(\rho_e u_e^* - \rho_w u_w^*\right)\Delta y + \left(\rho_n v_n^* - \rho_s v_s^*\right)\Delta x + \frac{(\rho_P^0 - \rho_P^{\Delta t})\Delta x\Delta y}{\Delta t}$$

The source term for pressure correction is the mass imbalance. Thus the correction will cease when the mass balance is satisfied.

2:48 PM 27/28

Details of the Procedure

1. Assume arbitrary u, v and p fields
2. Compute the coefficients for x and y momentum equations on lines given in Eq. (33). Proper labelling of indices have to be taken care of
3. Solve for u^* and v^* using computed coefficients in step 2 and assume pressure field in step 1
4. Compute mass imbalance (b) in Eq. (43).
5. If mass imbalance is low every where, then the obtained u^*, v^* and the assumed p^* are the solutions.

2:48 PM 28/28

Details of the Procedure (cont'd)

6. If step 5 is not satisfied, then obtain coefficients for pressure correction equation Eq. (43).
7. Solve for p', Eq. (43)
8. Correct u, v and p as, $p = p^* + \alpha_p p'$, $u = u^* + \alpha_u u'$ and $v = v^* + \alpha_v v'$, where α's are under relaxation coefficients to prevent the procedure from diverging. Typical values these are 0.5-0.7.
9. Repeat steps 2-8 as many times as is required to satisfy step 5.