ME-704 –CMTFE
Solution of Navier-Stokes Equations-III
MAC Method

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Introduction-I

- We had seen the stream function and vorticity formulation in our previous class
- Now we shall look at the finite difference formulation developed by the Los-Alamos National Lab in US.
- This method was originally called SOLA (some call it MAC) and has been subsequently modified for many specific cases including multi-phase flows.
- We shall restrict it for incompressible flows

Solution Method-I

- The conservative form of the incompressible NS equations in the absence of body forces and with uniform viscosity can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=-\frac{\partial \phi}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), where\ \phi=\frac{p}{\rho}$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}=-\frac{\partial \phi}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

- To generate an equation for pressure, we differentiate the momentum equations and add them

Solution Method-II

- This is done as follows

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right)$$

- To simplify matters, let us first look at LHS

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 v^2}{\partial y^2}+2\frac{\partial^2 uv}{\partial x\partial y}$$

$$\Rightarrow \frac{\partial D}{\partial t}+\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 v^2}{\partial y^2}+2\frac{\partial^2 uv}{\partial x\partial y},\quad where,\ D=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

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Solution Method-III

- Looking at RHS

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right)$$

$$\Rightarrow -\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\nu\left(\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right)$$

$$\Rightarrow \nu\left(\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)-\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$$

$$\Rightarrow \nu\left(\frac{\partial^2 D}{\partial x^2}+\frac{\partial^2 D}{\partial y^2}\right)-\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$$

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Solution Method-IV

- Thus the Poisson equation for pressure is

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 v^2}{\partial y^2}+2\frac{\partial^2 uv}{\partial x\partial y}=$$

$$\nu\left(\frac{\partial^2 D}{\partial x^2}+\frac{\partial^2 D}{\partial y^2}\right)-\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$$

$$\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=-\frac{\partial}{\partial t}(D)-\frac{\partial^2 u^2}{\partial x^2}-\frac{\partial^2 v^2}{\partial y^2}-2\frac{\partial^2 uv}{\partial x\partial y}+$$

$$\nu\left(\frac{\partial^2 D}{\partial x^2}+\frac{\partial^2 D}{\partial y^2}\right)$$

- Note that D has not been taken as 0 as this gives more accurate solution due to D not being zero exactly

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Solution Method-V

- Staggered concept is used to avoid checker boarding
- The method is explicit and uses simple central differencing.
- For a given initial velocity and pressure distribution, the velocities are advanced for one time step using FTCS method.
- Having obtained the velocities, new pressure distribution is obtained by solving the pressure Poisson equation. However, we need to derive the boundary conditions for ϕ
- Thus we have advanced the velocities and pressure by one time step.
- Similarly they are obtained for as many time steps as is desired.

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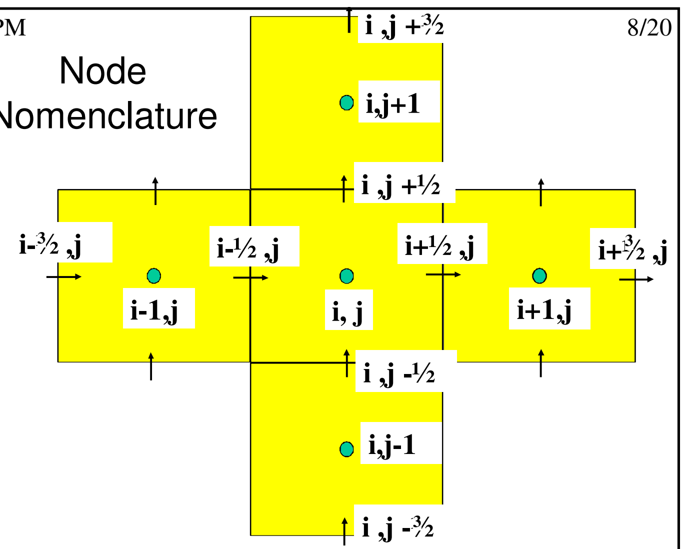
Node Nomenclature



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u-Velocity Update

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$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=-\frac{\partial \phi}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{u^{n+1}_{i+\frac{1}{2},j}-u_{i+\frac{1}{2},j}}{\delta t} = \frac{(u_{i,j})^2-(u_{i+1,j})^2}{\delta x}$$

$$+\ \frac{(u_{i+\frac{1}{2},j-\frac{1}{2}})(v_{i+\frac{1}{2},j-\frac{1}{2}})-(u_{i+\frac{1}{2},j+\frac{1}{2}})(v_{i+\frac{1}{2},j+\frac{1}{2}})}{\delta y}$$

$$+\ \frac{\varphi_{i,j}-\varphi_{i+1,j}}{\delta x}$$

$$+\ \frac{u_{i+\frac{3}{2},j}+u_{i-\frac{1}{2},j}-2u_{i+\frac{1}{2},j}}{\delta x^2}$$

$$+\ \frac{u_{i+\frac{1}{2},j+1}+u_{i+\frac{1}{2},j-1}-2u_{i+\frac{1}{2},j}}{\delta y^2}$$

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v-Velocity Update

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$$\frac{v^{n+1}_{i,j+\frac{1}{2}}-v_{i,j+\frac{1}{2}}}{\delta t} = \frac{(v_{i,j})^2-(v_{i,j+1})^2}{\delta y}$$

$$+\ \frac{(u_{i-\frac{1}{2},j+\frac{1}{2}})(v_{i-\frac{1}{2},j+\frac{1}{2}})-(u_{i+\frac{1}{2},j+\frac{1}{2}})(v_{i+\frac{1}{2},j+\frac{1}{2}})}{\delta x}$$

$$+\ \frac{\varphi_{i,j}-\varphi_{i,j+1}}{\delta y}$$

$$+\ \frac{v_{i+1,j+\frac{1}{2}}+v_{i-1,j+\frac{1}{2}}-2v_{i,j+\frac{1}{2}}}{\delta x^2}$$

$$+\ \frac{v_{i,j+\frac{3}{2}}+v_{i,j-\frac{1}{2}}-2v_{i,j+\frac{1}{2}}}{\delta y^2}.$$

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Interpolations

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$$u_{i+\frac{1}{2},j+\frac{1}{2}} \equiv \tfrac{1}{2}(u_{i+\frac{1}{2},j}+u_{i+\frac{1}{2},j+1}),$$

$$v_{i+\frac{1}{2},j+\frac{1}{2}} \equiv \tfrac{1}{2}(v_{i,j+\frac{1}{2}}+v_{i+1,j+\frac{1}{2}}),$$

$$u_{i,j} \equiv \tfrac{1}{2}(u_{i-\frac{1}{2},j}+u_{i+\frac{1}{2},j}),$$

$$v_{i,j} \equiv \tfrac{1}{2}(v_{i,j-\frac{1}{2}}+v_{i,j+\frac{1}{2}}).$$

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Discretised Pressure Poisson Eq.

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$$\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=-\frac{\partial}{\partial t}(D)-\frac{\partial^2 u^2}{\partial x^2}-\frac{\partial^2 v^2}{\partial y^2}-2\frac{\partial^2 uv}{\partial x\partial y}+\nu\left(\frac{\partial^2 D}{\partial x^2}+\frac{\partial^2 D}{\partial y^2}\right)$$

$$\frac{D^{n+1}_{i,j}-D_{i,j}}{\delta t} = -Q_{i,j}-\frac{\varphi_{i+1,j}+\varphi_{i-1,j}-2\varphi_{i,j}}{\delta x^2}$$

$$-\ \frac{\varphi_{i,j+1}+\varphi_{i,j-1}-2\varphi_{i,j}}{\delta y^2}$$

$$+\ \nu\left(\frac{D_{i+1,j}+D_{i-1,j}-2D_{i,j}}{\delta x^2}\right.$$

$$\left.+\ \frac{D_{i,j+1}+D_{i,j-1}-2D_{i,j}}{\delta y^2}\right),$$

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Source Term for Pressure Poisson Eq.

$$\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=-\frac{\partial}{\partial t}(D)-\frac{\partial^2 u^2}{\partial x^2}-\frac{\partial^2 v^2}{\partial y^2}-2\frac{\partial^2 uv}{\partial x\partial y}+\nu\left(\frac{\partial^2 D}{\partial x^2}+\frac{\partial^2 D}{\partial y^2}\right)$$

$$Q_{i,j} = \frac{(u_{i+1,j})^2 + (u_{i-1,j})^2 - 2(u_{i,j})^2}{\delta x^2} + \frac{(v_{i,j+1})^2 + (v_{i,j-1})^2 - 2(v_{i,j})^2}{\delta y^2} + \frac{2}{\delta x\ \delta y} \cdot \left[(u_{i+\frac{1}{2},j+\frac{1}{2}})(v_{i+\frac{1}{2},j+\frac{1}{2}}) + (u_{i-\frac{1}{2},j-\frac{1}{2}})(v_{i-\frac{1}{2},j-\frac{1}{2}}) - (u_{i+\frac{1}{2},j-\frac{1}{2}})(v_{i+\frac{1}{2},j-\frac{1}{2}}) - (u_{i-\frac{1}{2},j+\frac{1}{2}})(v_{i-\frac{1}{2},j+\frac{1}{2}})\right].$$

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Pressure Poisson Solution

$$\frac{\varphi_{i+1,j} + \varphi_{i-1,j} - 2\varphi_{i,j}}{\delta x^2} + \frac{\varphi_{i,j+1} + \varphi_{i,j-1} - 2\varphi_{i,j}}{\delta y^2} = -R_{i,j},$$

$$R_{i,j} \equiv Q_{i,j} - \frac{D_{i,j}}{\delta t} - \nu\left(\frac{D_{i+1,j} + D_{i-1,j} - 2D_{i,j}}{\delta x^2} + \frac{D_{i,j+1} + D_{i,j-1} - 2D_{i,j}}{\delta y^2}\right).$$

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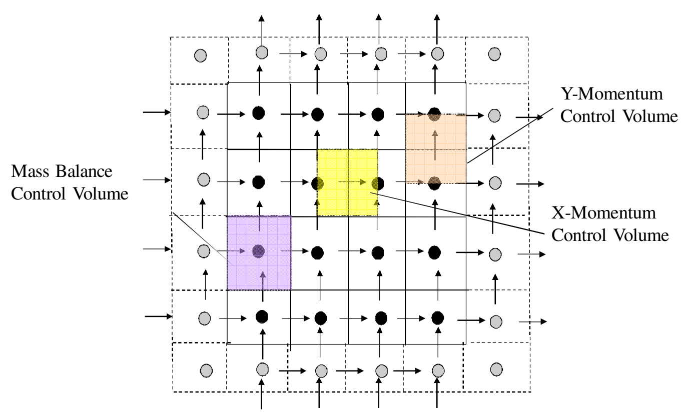
Solution Grid



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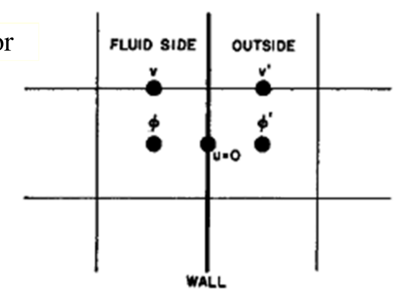
Boundary Conditions

- External velocities required in the ghost cells have to be specified using boundary conditions
- We shall restrict ourselves to no slip wall conditions.
- The external velocities are specified such that linear interpolations on the boundary shall satisfy the required boundary condition.
- This implies v' = -v for vertical wall.
- Similarly u' = -u for horizontal walls.
- We shall discuss the pressure boundary condition in next few slides

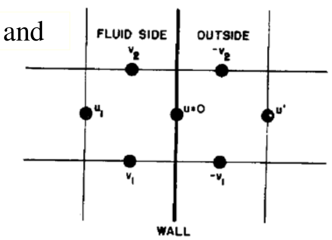


Pressure Solution

- Pressure will be solved for all the interior cells.
- Since it involves D of all the neighbours, D of the ghost cells have to be evaluated.
- This would imply, we need to specify velocities at the extreme boundaries of ghost cell.
- This is done by invoking the condition that D for ghost cell has to be zero
- Continuity for the fluid cell and ghost cell imply

$$\frac{0-u_1}{\Delta x}+\frac{v_2-v_1}{\Delta y}=0$$

$$\frac{u'-0}{\Delta x}+\frac{-(v_2-v_1)}{\Delta y}=0$$



- Addition of the two equations imply

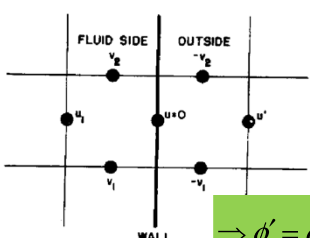
$$u'=u$$

- Thus all the exterior velocities are equal to the corresponding interior velocity
- The boundary condition for pressure for vertical walls can be obtained from the steady momentum equation as follows:

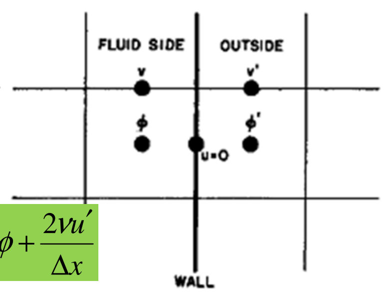
$$\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=-\frac{\partial \phi}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial \phi}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

- On the wall u, v and $\frac{\partial u}{\partial y}$ are all zero.
- This implies that $\frac{\partial^2 u}{\partial y^2}$ is also zero
- Hence $\frac{\partial \phi}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}$ $\Rightarrow \frac{\phi'-\phi}{\Delta x}=\frac{u'-0+u}{\Delta x^2}=\frac{2\nu u'}{\Delta x^2}$



$$\Rightarrow \phi'=\phi+\frac{2\nu u'}{\Delta x}$$



- Thus for right wall $\phi'=\phi+\frac{2\nu u'}{\Delta x}$
- For left wall $\phi'=\phi-\frac{2\nu u'}{\Delta x}$
- For top wall $\phi'=\phi+\frac{2\nu\ v'}{\Delta y}$
- For bottom wall $\phi'=\phi-\frac{2\nu\ v'}{\Delta y}$
- That completes all the necessary details