ME-704 –CMTFE-KNI

Introduction to Coordinate Transformations

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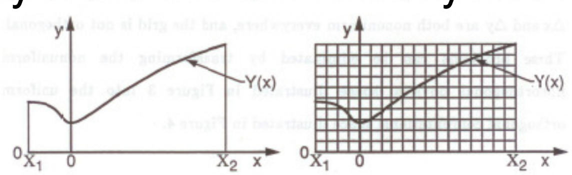
Body Fitted Coordinate Systems-I

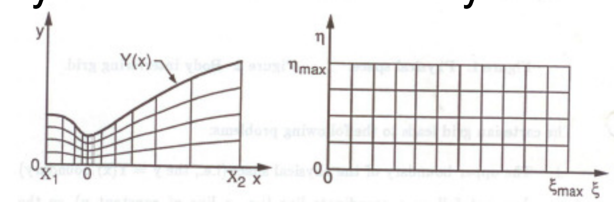
- We have seen a few methods to solve partial differential equations in Cartesian Domain
- However, in many practical instances, the systems are irregular and complex
- Solution in such cases would require special considerations
- If we use Cartesian system of equations, we get into difficulty of setting the boundary conditions accurately
- If we can develop body fitted coordinate systems, the substitution of boundary conditions become simple and accurate

Body Fitted Coordinate Systems-II



- Consider the domain of a converging-Diverging Nozzle as shown
- If we used a Cartesian system of coordinates, then representing the domain would require Staircasing
- The representation of Boundary conditions becomes complex and the boundary nodes have to be tagged
- Representing Neumann boundary conditions can be complex and inaccurate

Body Fitted Coordinate Systems-III



- Note that, if instead of x-y coordiante system, if we propose a coordinate system such that, ξ and η are constants along the boundaries
- In the computational space ξ and η are regular
- The representation of Boundary conditions becomes easy and accurate
- A classic case of boundary fitted coordinate system is the polar coordinate for cylinders (r-θ)

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Body Fitted Coordinate Systems-IV

- We know from our earlier study in heat transfer and fluid flow that the governing equations also have to be transformed for the modified coordinate systems and it will have additional terms
- For instance the governing equation

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=0$$

Transforms into

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}=0$$

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Body Fitted Coordinate Systems-V

- The transformed equations when solved in the new coordinate system would provide solutions in that coordinate system
- Thus, to locate the physical location of the point in the x-y coordinate system from the ξ and η coordinate system, relations will be needed
- Thus, we need three aspects
  1. Transform the governing equations and boundary conditions
  2. Obtain the solutions in the transformed domain
  3. Get the solution back in the physical domain

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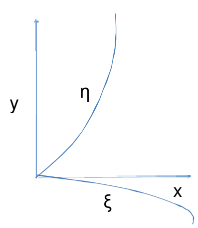
Transformation Rules-I

- Non-Orthogonal coordinate system is more general than the orthogonal ones and hence we will keep our discussion general
- Note that the transformation implies that ξ = ξ(x,y) and η = η (x,y)



- In general, for every (x,y) there should be a unique (ξ, η) and vice versa
- The transformations must be such that there will be continuity and no singularities
- Let us now look at transformation relations

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Transformation Rules-II

$$dx=\frac{\partial x}{\partial \xi}d\xi+\frac{\partial x}{\partial \eta}d\eta \qquad dy=\frac{\partial y}{\partial \xi}d\xi+\frac{\partial y}{\partial \eta}d\eta$$

$$\Rightarrow\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{bmatrix}\begin{Bmatrix}d\xi\\ d\eta\end{Bmatrix}=\begin{Bmatrix}dx\\ dy\end{Bmatrix}$$

- For a given dx and dy, dξ and dη will be unique, if

$$\begin{vmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{vmatrix}\neq 0$$

$$\Rightarrow I = x_\xi y_\eta - x_\eta y_\xi \neq 0$$

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Transformation Rules-III

- Similarly for inverse transformation , dξ and dη to dx and dy will be unique if

$$\Rightarrow J = \xi_x \eta_y - \xi_y \eta_x \neq 0$$

- We shall assume that these are satisfied and proceed further
- The transformation of Navier-Stokes equations are lengthy. But all the concepts can be understood just by considering the Laplace Equation.
- Let us look at the concepts now
- It is easy to verify that J = I^{-1} and vice versa

$$\Rightarrow J = \frac{1}{x_\xi y_\eta - x_\eta y_\xi}$$

We will use this in future

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Transformation Relations-I

- Let us consider $\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0$
- From Chain Rule,

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} = dy \qquad 1$$

- Since T also is a function of, ξ and η, we can write

$$dT = \frac{\partial T}{\partial \xi} d\xi + \frac{\partial T}{\partial \eta} d\eta$$

$$= \frac{\partial T}{\partial \xi}\left(\frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy\right) d\xi + \frac{\partial T}{\partial \eta}\left(\frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy\right)$$

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Transformation Relations-II

$$= \left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x}\right) dx + \left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y}\right) dy \qquad 2$$

- Comparing Eqs. (1) and (2) we can write,

$$\frac{\partial T}{\partial x} = \left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \quad and \quad \frac{\partial T}{\partial y} = \left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y}\right)$$

- Thus we can write the following transformation

$$\frac{\partial ()}{\partial x} = \left(\frac{\partial \xi}{\partial x}\frac{\partial ()}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial ()}{\partial \eta}\right) \quad and \quad \frac{\partial ()}{\partial y} = \left(\frac{\partial \xi}{\partial y}\frac{\partial ()}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial ()}{\partial \eta}\right)$$

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Transformation Relations-III

- We can proceed similarly and write

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\frac{\partial T}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial T}{\partial \eta}\right)$$

$$= \frac{\partial^2 \xi}{\partial x^2}\frac{\partial T}{\partial \xi} + \frac{\partial \xi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial \xi}\right) + \frac{\partial^2 \eta}{\partial x^2}\frac{\partial T}{\partial \eta} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial \eta}\right) \qquad 3$$

$$\frac{\partial \xi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial \xi}\right) = \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 T}{\partial \xi^2} + \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}\frac{\partial^2 T}{\partial \eta \partial \xi} \qquad 4$$

$$\frac{\partial \eta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial \eta}\right) = \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x}\frac{\partial^2 T}{\partial \eta \partial \xi} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 T}{\partial \eta^2} \qquad 5$$

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Transformation Relations-IV

- From Eqs. (3), (4) and (5), we can write

$$\frac{\partial^2 T}{\partial x^2}=\left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial^2 T}{\partial\xi^2}+2\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\left(\frac{\partial^2 T}{\partial\xi\partial\eta}\right)+\left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial^2 T}{\partial\eta^2}+\frac{\partial^2\xi}{\partial x^2}\frac{\partial T}{\partial\xi}+\frac{\partial^2\eta}{\partial x^2}\frac{\partial T}{\partial\eta}$$

- Proceeding similarly, we can write

$$\frac{\partial^2 T}{\partial y^2}=\left(\frac{\partial\xi}{\partial y}\right)^2\frac{\partial^2 T}{\partial\xi^2}+2\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\left(\frac{\partial^2 T}{\partial\xi\partial\eta}\right)+\left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial^2 T}{\partial\eta^2}+\frac{\partial^2\xi}{\partial y^2}\frac{\partial T}{\partial\xi}+\frac{\partial^2\eta}{\partial y^2}\frac{\partial T}{\partial\eta}$$

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Transformation Relations-V

- Thus we can write

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=a\frac{\partial^2 T}{\partial\xi^2}+\ b\frac{\partial^2 T}{\partial\xi\,\partial\eta}+c\frac{\partial^2 T}{\partial\eta^2}+d\frac{\partial T}{\partial\xi}+e\frac{\partial T}{\partial\eta}$$

- Where,

$$a=\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2=\nabla\xi.\nabla\xi,\qquad d=\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}=\nabla^2\xi,$$

$$b=2\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+2\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}=2\nabla\xi.\nabla\eta,\qquad e=\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}=\nabla^2\eta$$

$$c=\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2=\nabla\eta.\nabla\eta$$

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Illustrative Investigation-I

- Let us consider an illustrative example with algebraic transformation
- Consider the transformation

$$r=\xi=\sqrt{x^2+y^2},\quad \theta=\eta=\tan^{-1}\frac{y}{x}$$

- The inverse transformation relations are

$$x=rCos\theta,\quad y=rSin\theta$$

$$\frac{\partial\xi}{\partial x}=\frac{\partial(x^2+y^2)^{1/2}}{\partial x}=\frac{2x}{2(x^2+y^2)^{1/2}}=\frac{x}{r}=Cos\theta$$

$$\frac{\partial\xi}{\partial y}=\frac{\partial(x^2+y^2)^{1/2}}{\partial y}=\frac{2y}{2(x^2+y^2)^{1/2}}=\frac{y}{r}=Cos\theta$$

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Illustrative Investigation-II

$$\frac{\partial\eta}{\partial x}=\frac{Tan^{-1}(y/x)}{\partial x}=\frac{1}{(1+(y/x)^2}\left(-\frac{y}{x^2}\right)=\frac{-y}{x^2+y^2}=\frac{-y}{r^2}=\frac{-Sin\theta}{r}$$

$$\frac{\partial\eta}{\partial y}=\frac{Tan^{-1}(y/x)}{\partial x}=\frac{1}{(1+(y/x)^2}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}=\frac{x}{r^2}=\frac{Cos\theta}{r}$$

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Illustrative Investigation-III

$$\frac{\partial^2\xi}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial\xi}{\partial x}=\frac{\partial Cos\theta}{\partial x}=-Sin\theta\frac{\partial\theta}{\partial x}=-Sin\theta\frac{-Sin\theta}{r}=\frac{Sin^2\theta}{r}$$

$$\frac{\partial^2\xi}{\partial y^2}=\frac{\partial}{\partial y}\frac{\partial\xi}{\partial y}=\frac{\partial Sin\theta}{\partial y}=Cos\theta\frac{\partial\theta}{\partial y}=Cos\theta\frac{Cos\theta}{r}=\frac{Cos^2\theta}{r}$$

$$\frac{\partial^2\eta}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{-Sin\theta}{r}\right)=\frac{-1}{r}Cos\theta\frac{\partial\theta}{\partial x}+\frac{-1}{r^2}(-Sin\theta)\frac{\partial r}{\partial x}$$

$$=\frac{-Cos\theta}{r}\frac{-Sin\theta}{r}+\frac{(Sin\theta)}{r^2}Cos\theta=\frac{(2Sin\theta\ Cos\theta)}{r^2}$$

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Illustrative Investigation-IV

$$\frac{\partial^2\eta}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial\eta}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{Cos\theta}{r}\right)=\frac{1}{r}(-Sin\theta)\frac{\partial\theta}{\partial y}+\frac{-1}{r^2}(Cos\theta)\frac{\partial r}{\partial y}$$

$$=\frac{-Sin\theta}{r}\frac{Cos\theta}{r}+\frac{(-Cos\theta)}{r^2}Sin\theta=\frac{(-2Sin\theta\ Cos\theta)}{r^2}$$

$$a=\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2=Cos^2\theta+Sin^2\theta=1$$

$$0.5b=\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}=Cos\theta\frac{-Sin\theta}{r}+Sin\theta\frac{-Cos\theta}{r}=0$$

$$c=\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2=\frac{Sin^2\theta}{r^2}+\frac{Cos^2\theta}{r^2}=\frac{1}{r^2}$$

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Illustrative Investigation-V

$$d=\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}=\frac{Sin^2\theta}{r}+\frac{Cos^2\theta}{r}=\frac{1}{r}$$

$$e=\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}=\frac{2Sin\theta\ Cos\theta}{r^2}-\frac{2Sin\theta\ Cos\theta}{r^2}=0$$

- Thus we can write

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}+\frac{1}{r}\frac{\partial T}{\partial r}$$