ME-704 –CMTFE
Introduction to Concepts in Grid Generation

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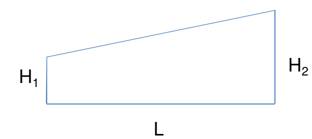
Grid Generation-I

- We had looked at the transformation of the governing equation
- If we have an algebraic transformation between ξ-η and x-y such as the one we saw in the last lecture the problem is straight forward
- For instance consider the simple case of a trapezoidal domain as shown
- We can define the following

$$\xi = \frac{x}{L}, \quad \eta = \frac{y}{y_t}$$

$$y_t = H_1 + \frac{H_2 - H_1}{L} x$$



The transformation makes the values of ξ and η go from 0 to 1 at the boundaries

Metric Relations-I

$$x = L\xi \quad y = \eta\left(H_1 + \frac{H_2 - H_1}{L} L\xi\right) = \eta\left(H_1 + (H_2 - H_1)\xi\right)$$

- The metric relations can be found out as

$$\xi_x = \frac{1}{L} \quad \xi_y = 0 \quad \xi_{xx} = 0 \quad \xi_{yy} = 0$$

$$\eta_x = -\frac{y}{\left(H_1 + \frac{H_2 - H_1}{L} x\right)^2} \frac{H_2 - H_1}{L}$$

Metric Relations-II

$$\eta_{xx} = 2\frac{y}{\left(H_1 + \frac{H_2 - H_1}{L} x\right)^3}\left(\frac{H_2 - H_1}{L}\right)^2$$

$$\eta_y = -\frac{1}{\left(H_1 + \frac{H_2 - H_1}{L} x\right)} \qquad \eta_{yy} = 0$$

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Metric Relations-III

- Thus we can compute a, b, c, d and e at every point By knowing the x and y at every (ξ,η) point in the computational plane.
- Now the solution can be obtained by appropriate finite difference formulation
- Often the inverse transformation is difficult to obtain and hence the entire grid generation is done numerically
- The metrics have also to be determined numerically. However, in these cases, rather than ξ_x, ξ_y, η_x and η_y we make use of x_ξ, x_η, y_ξ, y_η are used. We shall now relate these in the next slides

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Metric Relations-IV

$$dx = x_\xi d\xi + x_\eta d\eta \qquad dy = y_\xi d\xi + y_\eta d\eta$$

$$\Rightarrow \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \begin{Bmatrix} dx \\ dy \end{Bmatrix}$$

- We can compute dξ as follows

$$d\xi = \frac{\begin{vmatrix} dx & x_\eta \\ dy & y_\eta \end{vmatrix}}{\begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix}}$$

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Metric Relations-V

$$d\xi = \xi_x dx + \xi_y dy \quad \Rightarrow along \;\; dy = 0 \;\; \frac{d\xi}{dx} = \xi_x$$

$$\xi_x = \left.\frac{d\xi}{dx}\right|_{dy=0} = \frac{1}{dx}\frac{\begin{vmatrix} dx & x_\eta \\ 0 & y_\eta \end{vmatrix}}{\begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix}} = y_\eta J \qquad where \;\; J = \frac{1}{x_\xi y_\eta - x_\eta y_\xi}$$

$$\xi_y = \left.\frac{d\xi}{dy}\right|_{dx=0} = \frac{1}{dy}\frac{\begin{vmatrix} 0 & x_\eta \\ dy & y_\eta \end{vmatrix}}{\begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix}} = -x_\eta J$$

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Metric Relations-VI

- Similarly we can get

$$\eta_x = \left.\frac{d\eta}{dx}\right|_{dy=0} = \frac{1}{dy}\frac{\begin{vmatrix} x_\xi & dx \\ y_\xi & 0 \end{vmatrix}}{\begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix}} = -y_\xi J$$

$$\eta_y = \left.\frac{d\eta}{dy}\right|_{dx=0} = \frac{1}{dy}\frac{\begin{vmatrix} x_\xi & 0 \\ y_\xi & dy \end{vmatrix}}{\begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix}} = x_\xi J$$

- Thus we have established the relationships

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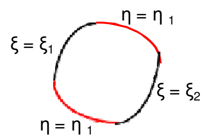
Grid Generation-I

- Our objective is to obtain a body fitted grid for an object whose boundary coordinates are known in terms of x and y
- We shall arbitrarily divide the body into four boundaries such as the one shown
- The objective is to determine metrics at every point as well as the relationship between x-y and ξ,η



- We have just seen that the metrics can be determined, if x_ξ, x_η, y_ξ, y_η are known at every (ξ,η) point.

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Governing Equations for Grids-I

- We are now going to seek to find some governing equations for x and y in the interior, whose solution will give us (x,y) for the known (ξ,η)
- In the previous lecture we had shown

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=a\frac{\partial^2 f}{\partial \xi^2}+b\frac{\partial^2 f}{\partial \xi\,\partial \eta}+c\frac{\partial^2 f}{\partial \eta^2}+d\frac{\partial f}{\partial \xi}+e\frac{\partial f}{\partial \eta}$$

- Where,

$$a=\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2 \qquad \Rightarrow a=\left(Jy_\eta\right)^2+\left(-Jx_\eta\right)^2$$

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Governing Equations for Grids-II

$$b=2\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \qquad \Rightarrow b=2\left(y_\xi y_\eta+x_\xi x_\eta\right)J^2$$

$$c=\left(\frac{\partial \eta}{\partial x}\right)^2+\left(\frac{\partial \eta}{\partial y}\right)^2 \qquad c=\left(x_\xi{}^2+y_\xi{}^2\right)J^2$$

$$d=\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 \xi}{\partial y^2}$$

$$e=\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 \eta}{\partial y^2}$$

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Governing Equations for Grids-III

- Some of the features that we desire in the grid mapping are
  - Continuity
  - Uniqueness
  - Maxima and Minima at the boundaries
- Solutions of Laplace equation satisfy these objectives
- Hence, it is very common to generate grids that satisfy Laplace equation. These are called elliptic grids.
- These grids are designed to satisfy

$$\nabla^2\xi=0,\ \ \nabla^2\eta=0$$

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Governing Equations for Grids-IV

- Thus from the previous slide, we can write

$$d = 0, \; e = 0$$

- If we substitute x for f we can get

$$\frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 x}{\partial y^2} = 0 = a\frac{\partial^2 x}{\partial \xi^2} + b\frac{\partial^2 x}{\partial \xi \partial \eta} + c\frac{\partial^2 x}{\partial \eta^2}$$

- Similarly, if we substitute y for x, we get

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0 = a\frac{\partial^2 y}{\partial \xi^2} + b\frac{\partial^2 y}{\partial \xi \partial \eta} + c\frac{\partial^2 y}{\partial \eta^2}$$

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Governing Equations for Grids-V

- Thus the final governing equations for x and y are

$$a\frac{\partial^2 x}{\partial \xi^2} + b\frac{\partial^2 x}{\partial \xi \partial \eta} + c\frac{\partial^2 x}{\partial \eta^2} = 0$$

$$a\frac{\partial^2 y}{\partial \xi^2} + b\frac{\partial^2 y}{\partial \xi \partial \eta} + c\frac{\partial^2 y}{\partial \eta^2} = 0$$

- The overall steps for solving for a complex domain can be summarised as follows

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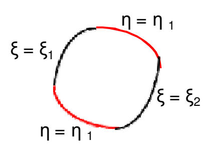
Steps for Solution-I

- Divide the object into four convenient boundaries
- The maxima and minima for ξ,η are set typically to 0 and 1
- Select NXN points on the boundaries. This can be done to the taste of the user. These can be uniformly or non-uniformly distributed
- The values of η on ξ = constant and the values of ξ on η = constant boundaries are now assigned
- The logical choice is to uniformly increase from 0 to 1.



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Steps for Solution-II

- For these (ξ,η) points on the boundaries, the physical values of (x,y) are assigned
- The values of (x,y) for the interior (ξ,η) points are guessed. Initial values can be the linearly interpolated values from the boundary
- The governing equations for x and y is solved as a boundary value problem simultaneously one iteration at a time. For this the values of a, b, c and J are estimated from the guessed values of x and y
- After one complete sweep of x and y, update the values of a, b and c. Under relaxation may be required to keep the iterations converge
- Iterate until convergence is obtained.

Steps for Solution-III

- Once the grid is solved for solve for T

$$a\frac{\partial^2 T}{\partial \xi^2} + 2b\left(\frac{\partial^2 T}{\partial \xi \partial \eta}\right) + c\frac{\partial^2 f}{\partial \eta^2} = 0$$

- The values of a, b and c are the ones already evaluated for the grid. Note d and e are 0 due to the choice of the grid.
- The boundary conditions for T will depend on the problem solved.