ME 704

Computational Methods in Thermal and Fluids Engineering

(Solution of Non-Linear Equations-2)

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Review

- We had seen the following methods
  - Bisection Method
  - Method of False Position
  - Secant Method
  - Newton's Method
  - Fixed Point Iteration
- We shall see the relative performances of three of these methods

Fixed Point Iteration-I

- Our recursive relation was

$$x_{n+1} = g(x_n) \qquad (1)$$

- If α is our root then

$$\alpha = g(\alpha) \qquad (2)$$

- Eqs. (1) and (2) imply that

$$x_{n+1} - \alpha = g(x_n) - g(\alpha) \qquad (3)$$

- Defining the error at any level i as

$$e_i = x_i - \alpha \qquad (4)$$

Fixed Point Iteration-II

- Eq. (3) can be written as

$$e_{n+1} = g(\alpha + e_n) - g(\alpha)$$

$$= \cancel{g(\alpha)} + e_n g'(\alpha) + \frac{e_n^2}{2} g''(\alpha) + \ldots - \cancel{g(\alpha)}$$

$$= e_n g'(\xi)$$ Using Mean Value Theorem

Where ξ is such that it lies between x_n and α

$$\Rightarrow \frac{e_{n+1}}{e_n} = g'(\xi)$$

Fixed Point Iteration-III

- For the method to converge,

$$\Rightarrow \left|\frac{e_{n+1}}{e_n}\right| < 1 \text{ or } |g'(\alpha)| < 1$$

- In fact this must be true in its entire path of initial guess all the way to the route, as otherwise, it can be thrown out anywhere
- Since $e_{n+1} = c\, e_n$ the method is said to have linear convergence near the root.
- It implies that the error will decrease linearly in the error-number of iteration plot

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Newton's Method-I

- In this case our recursive relation was

$$\Rightarrow x_{n+1} = x_n - f(x_n)/f'(x_n) \quad (1)$$

$$e_{n+1} + \alpha = e_n + \alpha - \frac{f(x_n) - f(\alpha)}{f'(x_n)}$$

Note that f(α) = 0 by definition

$$e_{n+1} + \cancel{\alpha} = e_n + \cancel{\alpha} - \frac{f(x_n) - f(x_n - e_n)}{f'(x_n)}$$

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Newton's Method-II

$$\Rightarrow e_{n+1} = e_n - \frac{f(x_n) - \left(f(x_n) + e_n f'(x_n) + (e_n^2/2) f''(x_n)\right)}{f'(x_n)}$$

$$\Rightarrow e_{n+1} = \frac{e_n^2\left(f''(x_n)\right)}{2f'(x_n)}$$

- For the method to converge,

$$\Rightarrow \left|\frac{e_{n+1}}{e_n}\right| < 1 \Rightarrow \left|\frac{e_n\left(f''(x_n)\right)}{2f'(x_n)}\right| < 1$$

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Secant Method-I

- The error analysis for this method is tedious but very illustrative of the power law technique
- In this case our recursive relation was

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

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Secant Method-II

$$\Rightarrow e_{n+1} + \alpha = e_n + \alpha - \frac{(x_n - x_{n-1}) f(\alpha + e_n)}{f(\alpha + e_n) - f(\alpha + e_{n-1})}$$

$$\Rightarrow e_{n+1} = e_n - \frac{(e_n - e_{n-1})(f(\alpha) + e_n f'(\alpha) + (e_n^2/2) f''(\alpha))}{f(\alpha) + e_n f'(\alpha) + (e_n^2/2) f''(\alpha) - (f(\alpha) + e_{n-1} f'(\alpha) + (e_{n-1}^2/2) f''(\alpha))}$$

= 0

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Secant Method-III

$$\Rightarrow e_{n+1} = e_n - \frac{(e_n - e_{n-1})(e_n f'(\alpha) + (e_n^2/2) f''(\alpha))}{(e_n - e_{n-1}) f'(\alpha) + \frac{e_n^2 - e_{n-1}^2}{2} f''(\alpha)}$$

$e_n + e_{n+1}$

$$\Rightarrow e_{n+1} = e_n - \frac{e_n + \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)}}{1 + \frac{e_n + e_{n-1}}{2} \frac{f''(\alpha)}{f'(\alpha)}}$$

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Secant Method-IV

- As x approaches the root,

$$\frac{e_n + e_{n-1}}{2} \frac{f''(\alpha)}{f'(\alpha)} << 1$$

$$\Rightarrow e_{n+1} = e_n - \left(e_n + \frac{e_n^2}{2} \frac{f''(\alpha)}{f''(\alpha)} \right) \left(1 - \frac{e_n + e_{n-1}}{2} \frac{f''(\alpha)}{f'(\alpha)} \right)$$

$$= e_n - \left(e_n - e_n \frac{e_n + e_{n-1}}{2} \frac{f''(\alpha)}{f'(\alpha)} + \frac{e_n^2}{2} \frac{f''(\alpha)}{f''(\alpha)} + O(e_n^3) \right)$$

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Secant Method-IV

$$\Rightarrow e_{n+1} = \frac{e_n e_{n-1}}{2} \frac{f''(\alpha)}{f'(\alpha)} \qquad (1)$$

- If we assume that the method is of order p, we can write

$$e_{n+1} = a e_n^p \text{ and } e_n = a e_{n-1}^p \Rightarrow e_{n-1} = \left(\frac{e_n}{a} \right)^{\frac{1}{p}}$$

- Eq.(1) can now be written as

$$\Rightarrow a e_n^p = \frac{e_n}{2} \left(\frac{e_n}{a} \right)^{\frac{1}{p}} \frac{f''(\alpha)}{f'(\alpha)}$$

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Secant Method-VI

- By reorganising terms, we get

$$\Rightarrow a^{1+\frac{1}{p}} e_n^{\,p} = \frac{1}{2} e_n^{1+\frac{1}{p}} \frac{f''(\alpha)}{f'(\alpha)}$$

- Since the power of n has to be homogeneous, we can write

$$p = 1 + \frac{1}{p} \qquad \Rightarrow p^2 - p - 1 = 0$$

$$\Rightarrow p = \frac{1 \pm \sqrt{5}}{2}$$

Secant Method-VII

- If $p < 1$, the method will diverge. Thus when the method converges, $p > 1$, which leads to

$$p = \frac{1+\sqrt{5}}{2} = 1.62$$

- Thus the method is inferior to Newton’s method, but needs only one function evaluation at a step and hence is competitive

Continuation Method

- Many times, the equation may be difficult to solve as the root is not known and the function is difficult
- A method called continuation method is very useful
- For an arbitrary x_0, we can say that x_0 is the root of the function $f(x)-f(x_0)$
- If we now define our function as $F(x) = f(x) - \beta f(x_0)$, and use x_0 as the guess for $\beta = 0.9$, we can find the root because the guess is good
- We can proceed in this manner successively by reducing β to 0, root of f(x) can be found