ME 704
Computational Methods in Thermal and Fluids Engineering
(Solution of Linear Equations-1)

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Review

- Understood that
  - Fixed point iteration method has linear convergence
  - Newton's method has quadratic convergence
  - Secant Method has the rate of convergence as 1.62
  - For difficult equations we can use continuation method to find the root

General-1

- These arise in
  - Network Analysis
  - Curve Fitting
  - Solution of ODE's and PDE's
- Its solution forms a basic part of most numerical algorithms

Matrix Notation

Consider a set of linear equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The above set may be represented by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Coefficient Matrix

Source vector

Types of Coefficient Matrices-I

Diagonal Matrix

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Tri-diagonal Matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$$

Types of Coefficient Matrices-II

Lower Diagonal Matrix [L]

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Upper Diagonal Matrix [U]

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Matrix Operations-I

Addition and Subtraction

$$A + B = C \Rightarrow a_{ij} + b_{ij} = c_{ij}$$

$$\begin{bmatrix} 3 & 6 & 2 \\ 4 & 1 & 7 \\ 8 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 6 & 4 \\ 3 & 7 & 1 \\ 6 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 12 & 6 \\ 7 & 8 & 8 \\ 14 & 6 & 6 \end{bmatrix}$$

Matrix Operations-II

Multiplication

$$A_{PXQ} \quad X \quad B_{QXR} = C_{PXR} \Rightarrow \sum_{k=1}^{Q} a_{ik} b_{kj} = c_{ij}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 16 & 13 & 10 \\ 47 & 40 & 33 & 26 \\ 75 & 64 & 53 & 42 \end{bmatrix}$$

Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$a_{11}(-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} +$$

$$a_{13}(-1)^4 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Towards Solution of Linear Equations-I

$$If \quad \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$Then \quad x_1 = \frac{b_1}{d_{11}} \quad x_2 = \frac{b_2}{d_{22}} \quad x_3 = \frac{b_3}{d_{33}}$$

$$Logic \Rightarrow For \quad i = 1, N \quad x_i = \frac{b_i}{d_{ii}}$$

Towards Solution of Linear Equations-II

$$If \quad \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$\Rightarrow \quad x_1 = \frac{b_1}{l_{11}}$$

$$x_2 = \frac{b_2 - l_{21}x_1}{l_{22}}$$

$$x_3 = \frac{b_3 - l_{31}x_1 - l_{32}x_2}{l_{33}}$$

$$Logic \Rightarrow$$

$$x_1 = \frac{b_1}{l_{11}}$$

$$For \quad i = 2, N$$

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij}x_j}{l_{ii}}$$

Towards Solution of Linear Equations-III

$$If \quad \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$\Rightarrow \quad x_3 = \frac{b_3}{u_{33}}$$

$$x_2 = \frac{b_2 - u_{23}x_3}{u_{22}}$$

$$x_1 = \frac{b_1 - u_{12}x_2 - u_{13}x_3}{u_{11}}$$

$$Logic \Rightarrow$$

$$x_N = \frac{b_N}{u_{NN}}$$

$$For \quad i = N-1,\ 1$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{N} u_{ij}x_j}{u_{ii}}$$