ME 704

Computational Methods in Thermal and Fluids Engineering

(Spline Interpolation)

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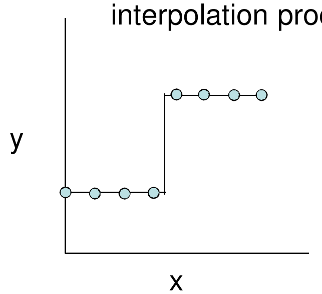
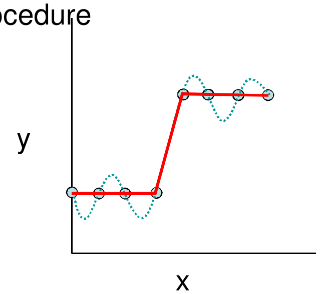
Review

- Interpolation is used to approximate the value of a function from a set of discrete values between two variables
- Looked at Lagrange Interpolation as the easiest polynomial interpolation
- Looked at Newton's backward and forward interpolating polynomials
- The values are obtained using difference tables
- Understood that a polynomial of N^{th} order has an error of the order of h^{N+1}

Spline Interpolation-I

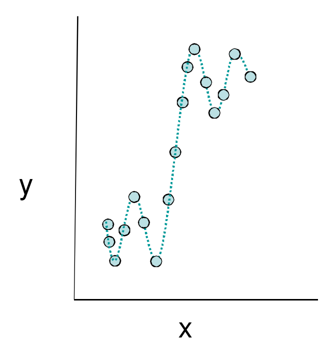
- It is one of the most powerful interpolation function
- Commonly used in graphics software
- It has all the characteristics of a good interpolation procedure





Spline Interpolation-II

- If the data is inherently wiggly, it will capture all twists and turns



Principle of Splines

- The concept of spline is to pass a specified order curve through a pair of points
- We know that we can pass a piece-wise linear curve between two pairs of points
- Can we pass a third order curve between every pair of points?
- The question is how?
- Quadratic spline fits a quadratic between every pair of point and cubic spline fits a cubic between every pair

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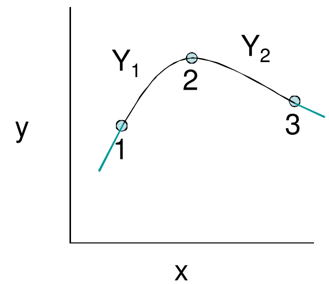
Cubic Spline

$Y_1 = a_1x^3 + b_1x^2 + c_1x + d_1$

$Y_2 = a_2x^3 + b_2x^2 + c_2x + d_2$



Conditions

$Y_1(x_1)=y_1$, $Y_1(x_2)=y_2$

$Y_2(x_2)=y_2$, $Y_2(x_3)=y_3$

$Y_1'(x_2) = Y_2'(x_2)$

$Y_1''(x_2) = Y_2''(x_2)$

$Y_1''(x_1) = Y_2''(x_3) = 0$

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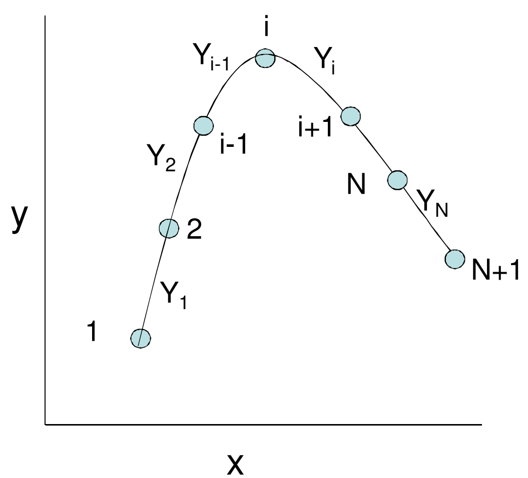

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Natural Cubic Spline-I

- In general if we have N+1 points, we shall have N intermediate curves. This will need 4N conditions
- For every interior point we shall have four conditions . Thus we shall have 4(N-1) conditions
- The functional values at the end points give two more conditions
- Assuming the second derivatives to be zero at the end points closes the relations for a natural spline.

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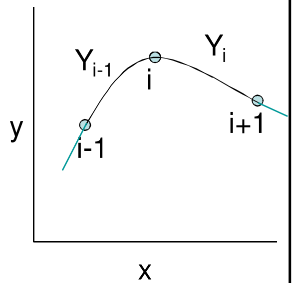
Natural Cubic Spline-II

- The curve $Y_i(x)$ joins i and i+1
- Since the curve is cubic, the second derivative $Y''_i(x)$ will b e a straight line
- Using Lagrange interpolation, we can write



$$Y_i''(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} Y_i''(x_i) + \frac{x - x_i}{x_{i+1} - x_i} Y_i''(x_{i+1})$$

Natural Cubic Spline-III

- Rewriting first term

$$Y_i''(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} Y_i''(x_i) + \frac{x - x_i}{x_{i+1} - x_i} Y_i''(x_{i+1}) \quad 1$$

$$Y_i''(x) = A_i \, Y_i''(x_i) + B_i \, Y_i''(x_{i+1})$$

Where,

$$A_i = \frac{x_{i+1} - x}{x_{i+1} - x_i}, \quad B_i = \frac{x - x_i}{x_{i+1} - x_i}$$

Note that,

$$1 - A_i = 1 - \frac{x_{i+1} - x}{x_{i+1} - x_i} = \frac{\cancel{x_{i+1}} - x_i - \cancel{x_{i+1}} + x}{x_{i+1} - x_i} = B_i$$

Natural Cubic Spline-IV

- Integrating Eq. (1)

$$Y_i'(x) = \frac{Y_i''(x_i)}{x_{i+1} - x_i} \frac{-(x_{i+1} - x)^2}{2} + \frac{Y_i''(x_{i+1})}{x_{i+1} - x_i} \frac{(x - x_i)^2}{2} + C_1 \quad 2$$

$$= \frac{-Y_i''(x_i) A_i (x_{i+1} - x)}{2} + \frac{Y_i''(x_{i+1}) B_i (x - x_i)}{2} + C_1$$

- Integrating Eq.(2), we get

$$Y_i(x) = \frac{Y_i''(x_i)}{x_{i+1} - x_i} \frac{(x_{i+1} - x)^3}{6} + \frac{Y_i''(x_{i+1})}{x_{i+1} - x_i} \frac{(x - x_i)^3}{6} + C_1 x + C_2$$

$$= \frac{Y_i''(x_i) A_i (x_{i+1} - x)^2}{6} + \frac{Y_i''(x_{i+1}) B_i (x - x_i)^2}{6} + C_1 x + C_2 \quad 3$$

Natural Cubic Spline-V

- Satisfying the functional values at terminal points
- $Y_i(x_i) = y_i$, $Y_i(x_{i+1}) = y_{i+1}$,
- Note that $A_i(x_i) = 1$, $B_i(x_i) = 0$, $A_i(x_{i+1}) = 0$, $Bi(x_{i+1}) = 1$,

$$y_i = \frac{Y_i''(x_i)}{x_{i+1} - x_i} \frac{(x_{i+1} - x_i)^2}{6} + C_1 x_i + C_2 \quad 4$$

$$y_{i+1} = \frac{Y_i''(x_{i+1})}{x_{i+1} - x_i} \frac{(x_{i+1} - x_i)^2}{6} + C_1 x_{i+1} + C_2 \quad 5$$

- Eq. (5)-Eq.(4)/$(x_{i+1}-x_i)$

$$C_1 = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{(Y_i''(x_{i+1}) - Y_i''(x_i))(x_{i+1} - x_i)}{6}$$

Natural Cubic Spline-VI

- Similarly, x_{i+1}Eq. (4) - x_iEq.(5)/(x_{i+1}-x_i)

$$C_2 = \frac{x_{i+1}y_i - x_i y_{i+1}}{x_{i+1} - x_i} - \frac{\left(x_{i+1}Y_i''(x_i) + x_i Y_i''(x_{i+1})\right)(x_{i+1} - x_i)}{6}$$

- Substitution of C1 and C2 in Eq. (3) and rearranging, we get

$$Y_i(x) = A_i y_i + B_i y_{i+1} + \begin{pmatrix} Y_i''(x_i)(A_i^3 - A_i) \\ + Y_i''(x_{i+1})(B_i^3 - B_i) \end{pmatrix} \frac{(x_{i+1} - x_i)^2}{6} \qquad 6$$

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Natural Cubic Spline-VII

- Now we shall match the first derivative at the interior point (x_i). From Slide 10, we have A_i and B_i

$$A_i = \frac{x_{i+1} - x}{x_{i+1} - x_i}, \quad B_i = \frac{x - x_i}{x_{i+1} - x_i} \qquad 6^+$$

$$\frac{dA_i}{dx} = \frac{-1}{x_{i+1} - x_i}, \quad \frac{dB_i}{dx} = \frac{1}{x_{i+1} - x_i}$$

- Further,

$$Y_i'(x) = \frac{dY_i}{dx} = \frac{dY_i}{dA_i}\frac{dA_i}{dx} = \frac{dY_i}{dB_i}\frac{dB_i}{dx}$$

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Natural Cubic Spline-VIII

- Differentiating Eq. (6) with x, we have

$$Y_i'(x) = \frac{dA_i}{dx} y_i + \frac{dB_i}{dx} y_{i+1} + \begin{pmatrix} Y_i''(x_i)(3A_i^2 - 1)\dfrac{dA_i}{dx} \\ + Y_i''(x_{i+1})(3B_i^2 - 1)\dfrac{dB_i}{dx} \end{pmatrix} \frac{(x_{i+1} - x_i)^2}{6}$$

$$= -\frac{y_i}{x_{i+1} - x_i} + \frac{y_{i+1}}{x_{i+1} - x_i} + \begin{pmatrix} -Y_i''(x_i)\dfrac{(3A_i^2 - 1)}{x_{i+1} - x_i} \\ + Y_i''(x_{i+1})\dfrac{(3B_i^2 - 1)}{x_{i+1} - x_i} \end{pmatrix} \frac{(x_{i+1} - x_i)^2}{6}$$

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Natural Cubic Spline-IX

- Note that $Y_i'(x_i) = Y_{i-1}'(x_i)$
- Further, $A_i(x_i) = 1$, $B_i(x_i) = 0$,

$$Y_i'(x_i) = -\frac{y_i}{x_{i+1} - x_i} + \frac{y_{i+1}}{x_{i+1} - x_i} + \begin{pmatrix} \dfrac{-2Y_i''(x_i)}{x_{i+1} - x_i} \\ -\dfrac{Y_i''(x_{i+1})}{x_{i+1} - x_i} \end{pmatrix} \frac{(x_{i+1} - x_i)^2}{6} \qquad 7$$

- Changing i to i-1, $A_{i-1}(x_i) = 0$, $B_{i-1}(x_i) = 1$,

$$Y_{i-1}'(x_i) = -\frac{y_{i-1}}{x_i - x_{i-1}} + \frac{y_i}{x_i - x_{i-1}} + \begin{pmatrix} \dfrac{Y_{i-1}''(x_{i-1})}{x_i - x_{i-1}} \\ -\dfrac{2Y_{i-1}''(x_i)}{x_i - x_{i-1}} \end{pmatrix} \frac{(x_i - x_{i-1})^2}{6} \qquad 8$$

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Natural Cubic Spline-X

- Eqs. (7) and (8) are rewritten as

$$Y_i'(x_i) = \frac{-y_i + y_{i+1}}{x_{i+1} - x_i} + \left(-2Y_i''(x_i) - Y_i''(x_{i+1})\right)\frac{(x_{i+1} - x_i)}{6} \quad (9)$$

$$Y_{i-1}'(x_i) = \frac{-y_{i-1} + y_i}{x_i - x_{i-1}} + \left(Y_{i-1}''(x_{i-1}) - 2Y_{i-1}''(x_i)\right)\frac{(x_i - x_{i-1})}{6} \quad (10)$$

- Eqs. (9) and (10) and rearranging (note that Y" is unique at the nodal points

$$Y''(x_{i-1})\frac{(x_i - x_{i-1})}{6} + 2Y''(x_i)\frac{(x_{i+1} - x_{i-1})}{6} + Y''(x_{i+1})\frac{(x_{i+1} - x_i)}{6} = \frac{(y_{i+1} - y_i)}{(x_{i+1} - x_i)} - \frac{(y_i - y_{i-1})}{(x_i - x_{i-1})} \quad (11)$$

Natural Cubic Spline-XI

- The above relation is valid for I = 2, 3, …, N
- For natural spline, Y" (1), Y" (N+1) = 0
- Eq. (10) is a list set of N-1 eqs., for Y" (2), Y" (3), …, Y" (N).
- Once these are computed, the value of y is found using Eq. (6). The A_i and B_i are found using Eq. (6^+)

Algorithm

- Input x(i), y(i) for i = 1, 2, 3, …, N, N+1
- Formulate TDMA Coefficients and RHS. (Refer Eq. (11)
- For natural spline, Y" (1), Y" (N+1) = 0
- Solve for Y" (2) to Y" (N) Using TDMA
- Search for the interval where the interpolation is required
- Once these are computed, the value of y is found using Eq. (6). The A_i and B_i are found using Eq. (6^+)