Assignment 7

1. Classify the following set as hyperbolic or elliptic.

$$a1\frac{\partial f}{\partial t} + a2\frac{\partial f}{\partial x} + a3\frac{\partial g}{\partial t} + a4\frac{\partial g}{\partial x} = a5$$
$$b1\frac{\partial f}{\partial t} + b2\frac{\partial f}{\partial x} + b3\frac{\partial g}{\partial t} + b4\frac{\partial g}{\partial x} = b5$$

In the class we showed that the characteristic directions were obtained by equating the determinant of the matrix [B-λA] = 0. Note that the expansion of the determinant will give you a quadratic in λ = dx/dt. Perform the expansion of the determinant and get the coefficients of the quadratic equation in terms of the coefficients of the governing equations, viz., a1, a2, a3, a4, b1, b2, b3 and b4.

The same can be shown in a long drawn way. This can be done in a similar way to the method adopted for one variable T that we did in the class. It involves the following steps. (1) Write 4 equations; 2 governing equations, one chain rule for df and another chain rule for dg. (2) Now write these 4 equations in the matrix form [A]{v}={R}, where [A] is a 4 x 4 matrix, {v} is a vector consisting of f_t, f_x, g_t, g_x as components and {R} is right hand side vector. (3) Now for discontinuities to exist, take the determinant [A] and equate it to 0. As usual expand the determinant. This will require taking cofactors, etc. You can now see that the expanded determinant will be nothing but a quadratic equation in λ, which is identical to the first case

2. Consider the convection equation

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$$

It is proposed to use the following finite difference approximations (FTCS):

$$f_t = \frac{f_i^{n+1} - f_i^{n+1}}{\Delta t}, \quad f_x = \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta x}$$

Using the above, perform consistency analysis and obtain the modified partial differential equation.

(a) Show that the method is O (Δt, Δx^2),

(b) Perform Von Neumann stability analysis and obtain G to be 1− IC sinθ and hence it is unconditionally unstable.

(c) Convert all the higher order time derivatives into space derivatives using the governing the governing differential equation. While doing so carry terms upto fxxxx. Show the final MPDE is of the form

$$T_t + uT_x = -\frac{1}{2}u^2\Delta t T_{xx} - \left(\frac{1}{6}u\Delta x^2 + \frac{1}{3}u^3\Delta t^2\right)T_{xxx} + \text{HOT}$$

3. Show the relationship

$$c_m = \frac{a_m - Ib_m}{2}, \quad +c_{-m} = \frac{a_m + Ib_m}{2}$$

This can be done by expanding the second equation and equating the relevant terms with the first equation in slide 31 of the lecture

4 Consider the diffusion equation

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

It is proposed to use the following finite difference approximations:

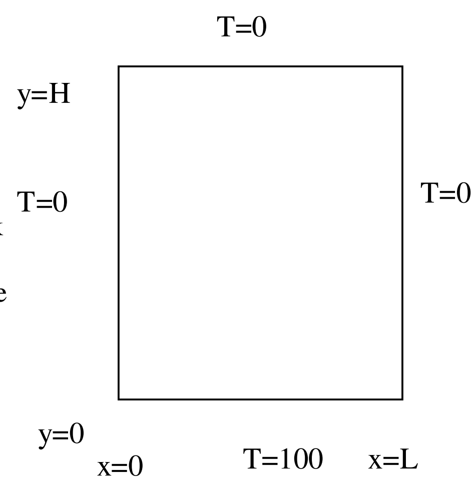
$$f_t = \frac{f_i^{n+1} - f_i^{n+1}}{\Delta t},\ f_{xx} = 0.5\frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\Delta x)^2} + 0.5\frac{f_{i+1}^{n} - 2f_i^{n} + f_{i-1}^{n}}{(\Delta x)^2},$$

Using the above, perform consistency analysis and obtain the modified partial differential equation. Convert all the higher order time derivatives into space derivatives using the governing the governing differential equations. By doing so carry terms upto f_{xxxx}.

(a) Show that the method is $O((\Delta t)^2, (\Delta x)^2)$, (Be patient with your algebra)

(b) Perform Von Neumann stability analysis and obtain G to be $\frac{1-2d(\sin\frac{\theta}{2})^2}{1+2d(\sin\frac{\theta}{2})^2}$ and hence it is unconditionally stable.

5. Consider the steady diffusion equation



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

in the domain as shown. The particular problem chosen is from the book Hoffman and Chiang. The domain of the plate and the boundary conditions are shown in the figure. The value of H = 2 and L = 1. The convergence criterion applied by the authors is

$$error = \sum_{\substack{1=2\\ j=2}}^{\substack{i=i\max\\ j=j\max}} abs(T_{i,j}^{k+1} - T_{i,j}^{k}) \le 0.01$$

The analytical solution for the problem is

$$T(x,y) = 100\left(2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\frac{Sinh\left(\frac{n\pi(H-y)}{L}\right)}{Sinh\left(\frac{n\pi H}{L}\right)}Sin\left(\frac{n\pi x}{L}\right)\right)$$

(a) Solve this problem by PSOR with values of Over-relaxation parameter of 1.8. Compare the variation of the centreline temperature along y, T(0.5,y), with the analytical solution. Use 20 terms for the analytical solution.
(b) Now run the program with relaxation parameter values of 1, 1.1, 1.2, 1.4, 1.6, 1.7, 1.8, 1.9 1.95 and 2.0. Plot the number of iterations as a function of the relaxation parameter. Discuss.
(c) Repeat the same with Line SOR method discussed in the class and find the optimum value of ω for this method. Report your results. The results should be similar to what we gave in the class.