Assignment 8

Note: Since the assignment has been posted late, everyone will be expected to solve Problem 1 and 2. Those who solve 3, will be given bonus points.

1. Consider the diffusion equation

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

Use the θ method described in the class (Slide 16/26 onwards).

(a) Perform Consistency analysis and show that you can arrive at the equation given in slide 17. (Be patient with your algebra)

(b) Now show that the method is $O((\Delta t)^2, (\Delta x)^2)$, when $\theta = \frac{1}{2}$

(c) Proceed to show that the method is $O((\Delta t)^2, (\Delta x)^4)$, when $\theta = \frac{1}{2} - \frac{(\Delta x)^2}{12\alpha\Delta t}$

(d) Proceed to show that the method is $O((\Delta t)^2, (\Delta x)^6)$, when

$$\theta = \left(\frac{1}{2} - \frac{\Delta x^2}{12\alpha\Delta t}\right) \text{ and } \frac{\alpha\Delta t}{\Delta x^2} = \frac{1}{\sqrt{20}}$$

Please note that the expression in slide 17 has considered expansion only up to third derivative of time and sixth derivative with respect to space. In order to show the above, one has to go up to eighth derivative in space and fourth derivative in time when expansion is done. Extra points will be awarded to those who have patience to do this. Others can just verify that by taking the $\alpha^3\Delta t^2$ common from third term in square bracket, and substituting $\theta = \frac{1}{2} - \frac{1}{12D}$, where $D = \frac{\alpha\Delta t}{(\Delta x)^2}$, the entire term will vanish for the conditions stated.

(e) Perform Von Neumann stability analysis and obtain G and verify that the method is unconditionally stable for $1 \geq \theta \geq \frac{1}{2}$ and has conditional stability $0 \leq D \leq \frac{1}{2-4\theta}$

2. Assume that you are solving for the diffusion equation in the domain t>0, 0<x<1 with the following initial and boundary conditions

$f(x,0) = \sin(\pi x)$, $f(0,t) = 0 = f(1,t)$. Show that the analytical solution is

$f(x,t) = \exp(-\alpha\pi^2 t)\sin(\pi x)$ (This you can show by mere substitution.

Write a computer code to solve the diffusion equation using the θ method. Normally, while coding you will use TDMA as the solution method for $\theta > 0$. For $\theta = 0$, the solution can be explicitly obtained. Thus, you need to have both parts separately coded with suitable options that will direct the program to the relevant part. Since in the present assignment, I am not asking you to solve for $\theta = 0$, you can skip this explicit part to save time.

Take the value of alpha = 0.01and delx = 0.1. Do as directed. Carry all calculations upto t = 30 or just close to it.

(i) Solve for $\theta = 0.25$, with $\Delta t = 0.8$ and 1.2. Comment on the results.

(ii) Solve for theta = 0.5, and theta = 1. with $\frac{\alpha\Delta t}{(\Delta x)^2} = 5$ (choose Δt to satisfy this)

(iii) $\frac{\alpha\Delta t}{(\Delta x)^2} = \frac{1}{\sqrt{20}}$ and $\theta = \frac{1}{2} - \frac{\sqrt{20}}{12}$

(iv) $\dfrac{\alpha\Delta t}{(\Delta x)^2} = \dfrac{1}{\sqrt{20}}$ and $\theta = \dfrac{1}{2}$

Comment on the accuracy and nature of error (always negative, oscillating between negative and positive, etc.)

3. Now let us extend the problem to 2-D.

$$\frac{\partial f}{\partial t} = \alpha\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

Assume that you are solving for the diffusion equation in the domain
$t>0$, $0<x<1$ and $0<y<1$ with the following initial and boundary conditions
$f(x,y,0) = \sin(\pi x)\sin(\pi y)$
$f(0,y,t) = f(1,y,t) = f(x,0,t) = f(x,1,t) = 0$

(a) Show that the analytical solution is
$f(x,y,t) = \exp(-2\alpha\pi^2 t)\sin(\pi x)\sin(\pi y)$

(b) Now program the fractional step method given using a kind of ADI with delx = 0.1, dely = 0.1, alpha = 0.01, delt = 0.1 and Compare the solution with the analytical solution.

(c) Numerically verify that the method is unconditionally stable.

*******************HAPPY PROGRAMMING*********************