Mechanics Review-III

- Elasticity
- Material Models
- Yielding criteria, Tresca and Von Mises
- Levy-Mises equations
- Case Study for multiple failure mechanism

Prof. Ramesh Singh

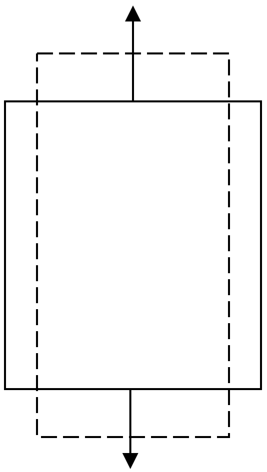
Elastic Stress-Strain

- Linear stress-strain

$$\sigma_x = E\varepsilon_x$$

- The extension in one direction is accompanied by contraction in other two directions, for isotropic material

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x$$



Hooke's Law

- In x direction, strain produced by stresses

Strains in x direction due to various stresses

$$\sigma_x \rightarrow \frac{\sigma_x}{E}$$

$$\sigma_y \rightarrow -\frac{\nu\sigma_y}{E}$$

$$\sigma_z \rightarrow -\frac{\nu\sigma_z}{E}$$

Superimposing,

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}\left(\sigma_y + \sigma_z\right)$$

Hooke's Law

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}\left(\sigma_y + \sigma_z\right)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}\left(\sigma_x + \sigma_z\right)$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$$

$$\tau_{xy} = G\gamma_{xy}$$

$$\tau_{yz} = G\gamma_{yz}$$

$$\tau_{xz} = G\gamma_{xz}$$

Stiffness Matrix

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}$$

Compliance Matrix

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

Poisson's Ratio

- Adding the strains

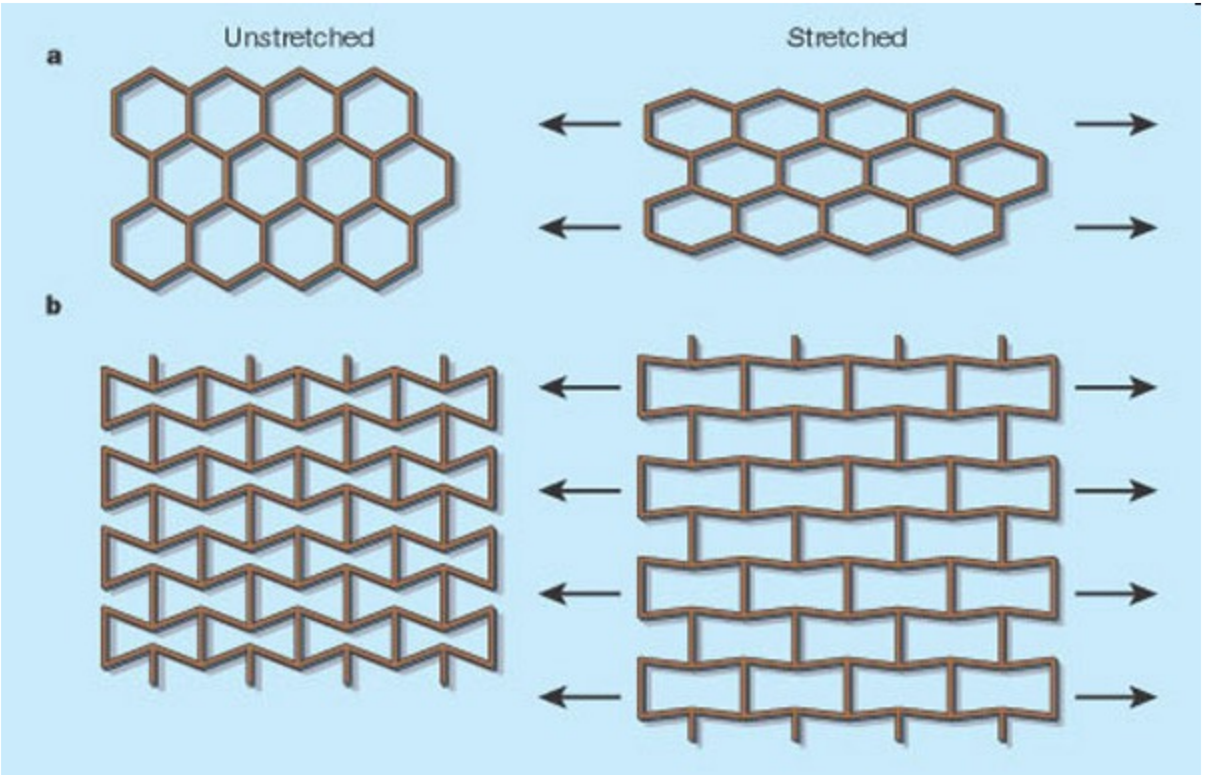
$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}\left(\sigma_y + \sigma_y + \sigma_z\right)$$

$$\frac{\Delta V}{V} = e_x + e_y + e_z, \text{Engineering strains for elastic condition}$$

- For fully plastic,

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = 0$$

$\nu = 0.5$, Typically, $-1<\nu<0.5$, Most metals, 0.3

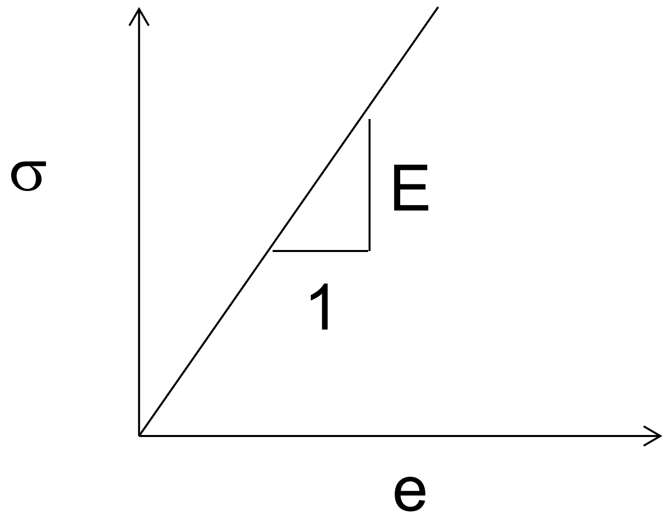


Stretching these two-dimensional hexagonal structures horizontally reveals the physical origin of Poisson's ratio. **a,** The cells of regular honeycomb or hexagonal crystals elongate and narrow when stretched, causing lateral contraction and so a positive Poisson's ratio. **b,** In artificial honeycomb with inverted cells, the structural elements unfold, causing lateral expansion and a negative Poisson's ratio.

Constitutive Behavior Equations

Linear elastic (simplest)

- Young's modulus, $E = \sigma/e$

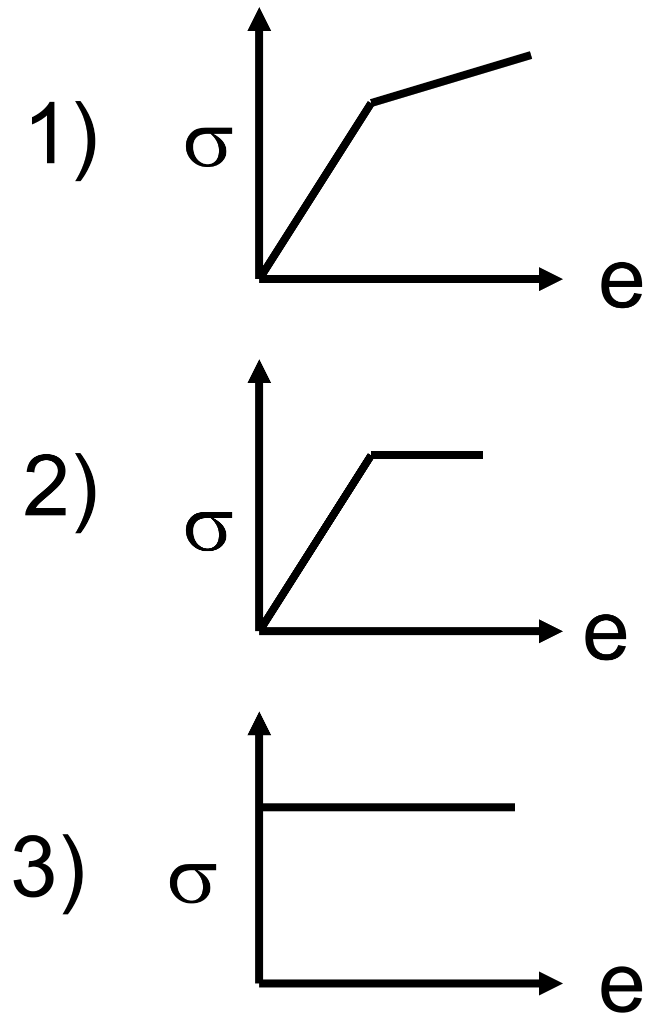


Thomas Young
1773-1829

Robert Hooke
1635-1703

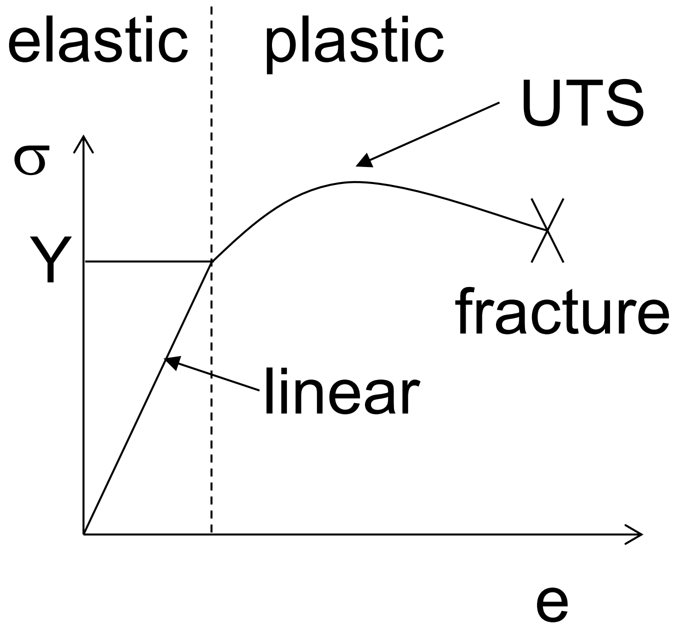
#2 - Match

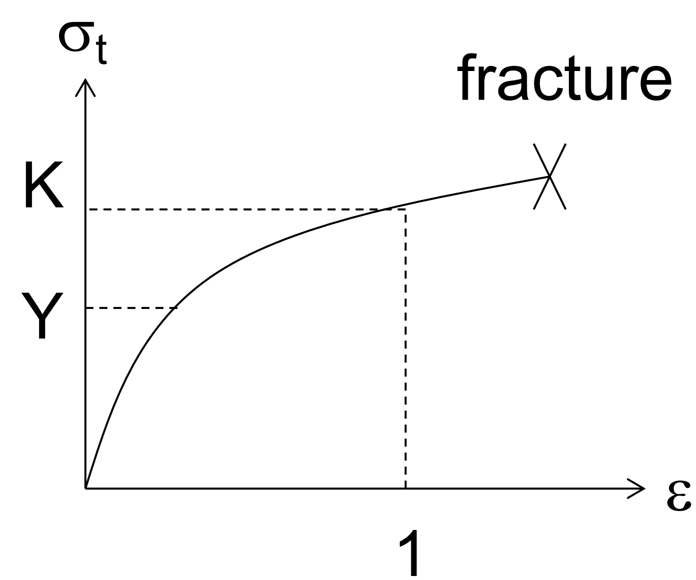
a) Elastic – plastic material
b) Perfectly plastic material
c) Elastic – linear strain hardening material



Actual Material Behavior

$$\sigma_t = K\varepsilon^n$$





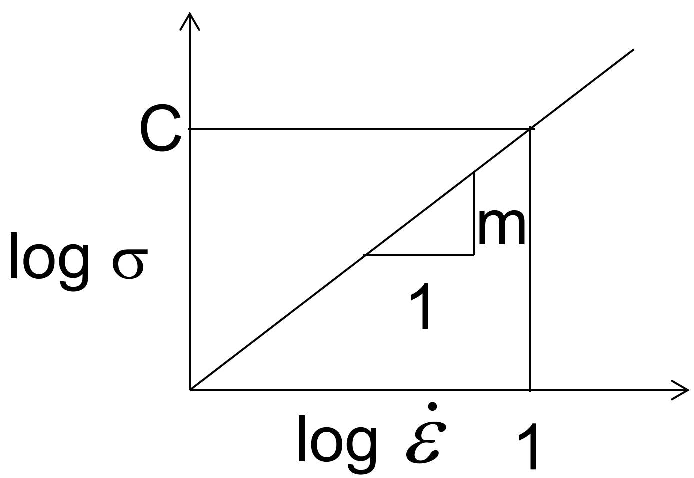
K = strength coefficient
n = strain hardening coefficient

Strain Rate Effect (1)

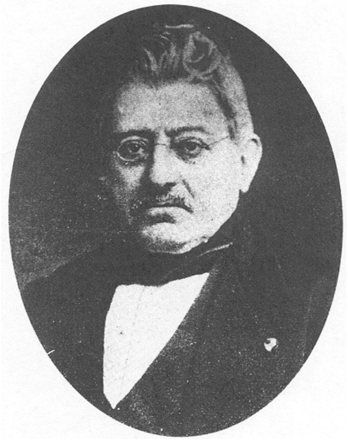
$$\sigma = C\dot{\varepsilon}^m$$

C = strength coefficient

m = strain rate sensitivity coefficient



Yield Criteria



Henri Tresca
1814-1885

- How do you know if a material will fail?
  Compare loading to various yield criteria
  - Tresca
  - von Mises

Richard von Mises
1883-1953

- Key concepts
  - plane stress
  - plane strain

#3 - Match

a) Tresca yield criterion

b) Von Mises yield criterion

c) Maximum distortion energy criterion

d) Maximum shear stress criterion

1) $\tau_{max} \geq \frac{1}{2} \sigma_{yield}$

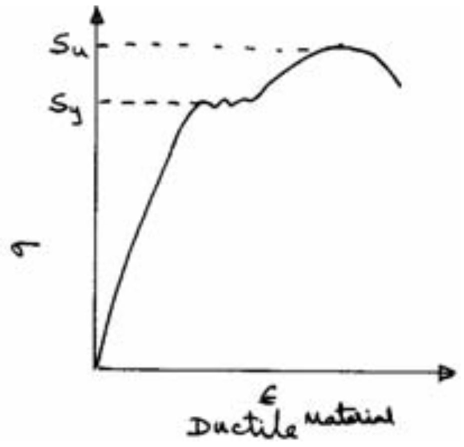
2) $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2$

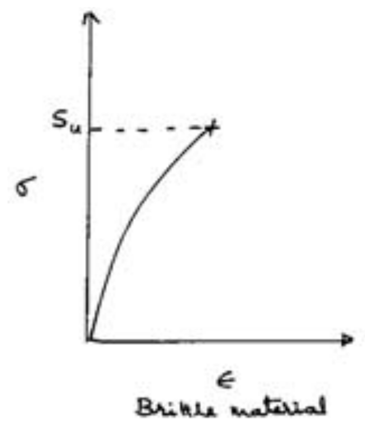
3) $\sigma_3-\sigma_1 = 2\tau_{yield}$

Stress-Strain Relationship

- Strength of a material or failure of the material is determined from uniaxial stress-strain tests

- The typical stress-strain curves for ductile and brittle materials are shown below.





Need for a failure theory

- In the case of multiaxial stress at a point we have a more complicated situation present. Since it is impractical to test every material and every combination of stresses σ_1, σ_2, andσ_3, a failure theory is needed for making predictions on the basis of a material's performance on the tensile test.

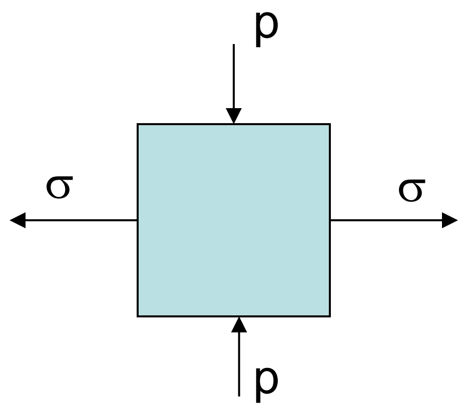
Importance of Yield Criterion

Assume three loading conditions in plane stress:

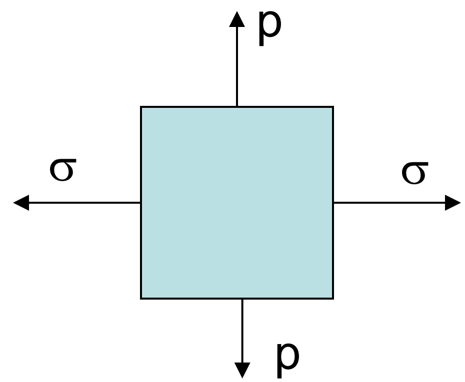
$$\sigma_1 - \sigma_3 = Y$$

At the onset of yielding,

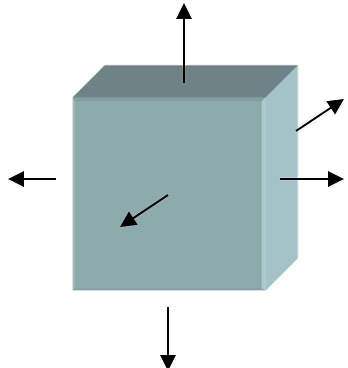
$$\sigma = Y$$



$$\sigma + p = Y$$



$$\sigma - p = Y$$



Tresca Yield Criterion

$\tau_{max} \geq k$ or τ_{flow} (shear yield stress) a material property

$\sigma_{max} - \sigma_{min} = Y = 2k = 2\tau_{flow}$

$$\tau = \frac{\sigma_{max} - \sigma_{min}}{2} = k$$

- Simple tension

  $\sigma_1 = Y = 2k = 2\tau_{flow}$

- Under plane stress

  $\sigma_1 - \sigma_3 = Y$

Derivation of Distortion Energy

- Done in class

Von Mises Yield Criterion

Based on Distortion Energy

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2$$

$$\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right) = 2Y^2$$

Y = uni-axial yield stress

VonMises/Tresca Criterion

- The locus of yielding for VonMises

$\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 = Y^2$

Using Tresca Criterion in first quadrant

$\sigma_1 > 0, \sigma_3 > 0$

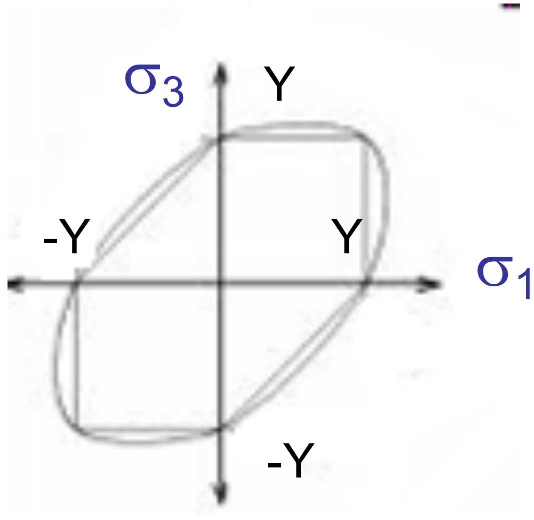
$\sigma_2 = 0$

for plane stress maximum value of $\sigma_1, \sigma_3 = Y$ in first and third quadrant

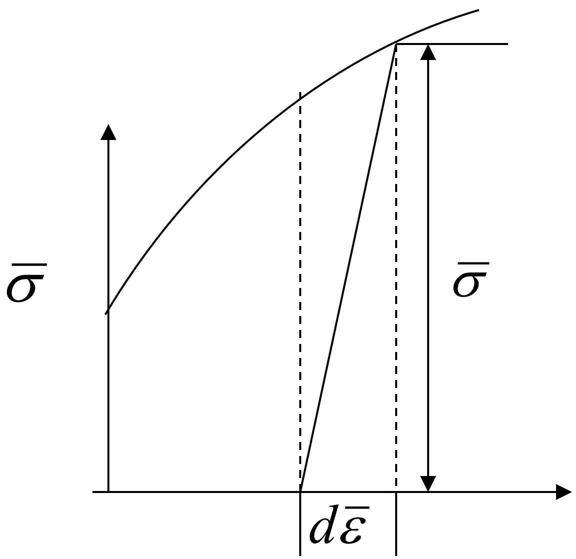
second and fourth it will be a 45 deg line,

$\sigma_1 - \sigma_3 = Y$

Locus of Yield



Levy Mises Flow Eqn.



$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}\left(\sigma_y + \sigma_z\right)$$

Analogous to Elastic Equation

$$d\varepsilon_1 = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_1 - \frac{1}{2}\left(\sigma_2 + \sigma_3\right)\right]$$

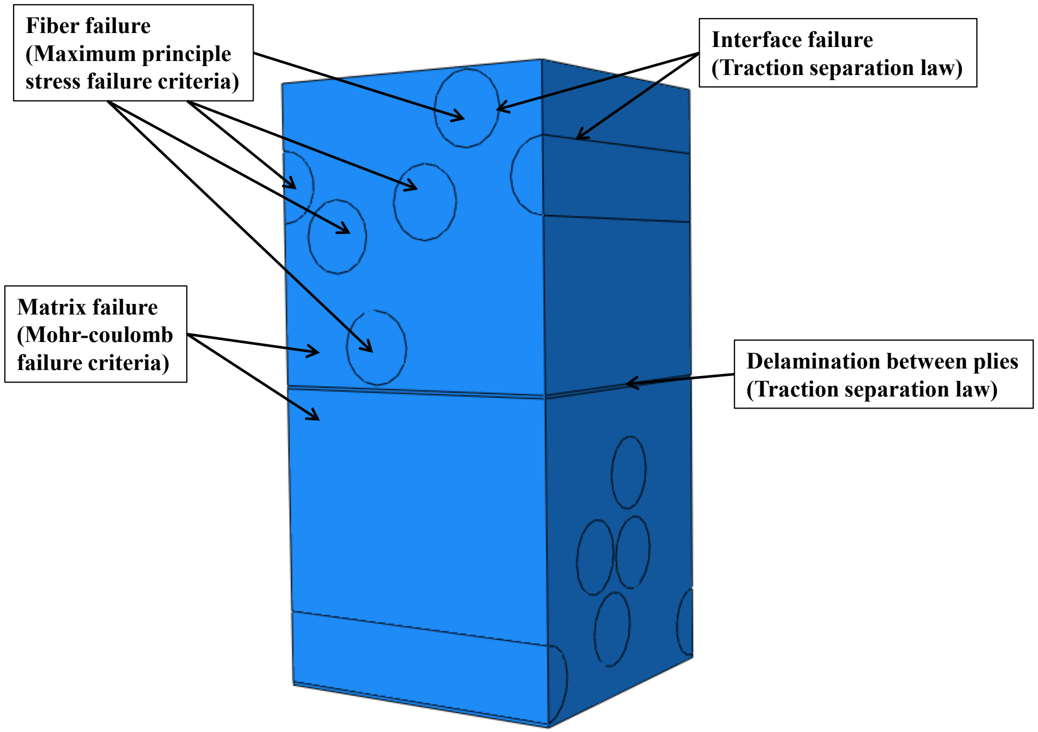
$$d\varepsilon_2 = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_2 - \frac{1}{2}\left(\sigma_1 + \sigma_3\right)\right]$$

$$d\varepsilon_3 = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_3 - \frac{1}{2}\left(\sigma_1 + \sigma_2\right)\right]$$

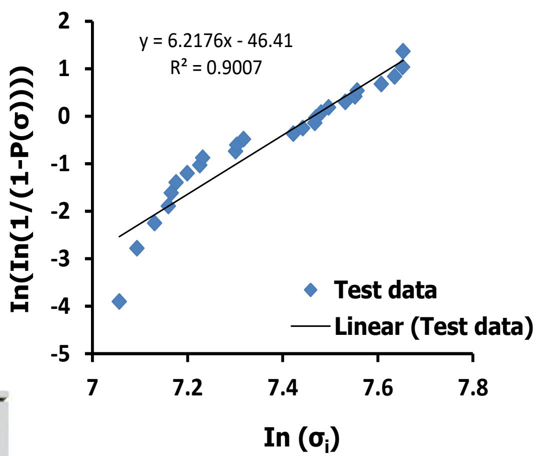
For Brittle Materials

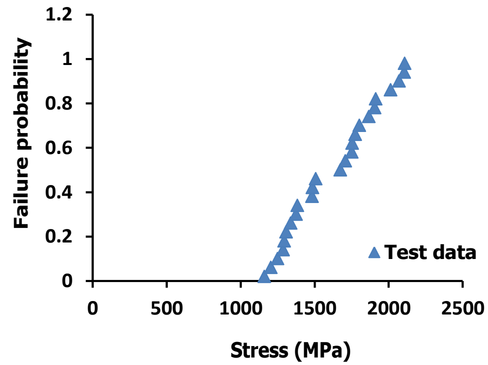
- Brittle Materials
  - Hardened steels exhibit symmetry tension and compression so preferred failure is maximum normal stresses or principal stresses
  - Some materials which exhibit tension compression asymmetry such cast iron need a different failure theory such Mohr Coulomb

Failure criteria a case study in composites



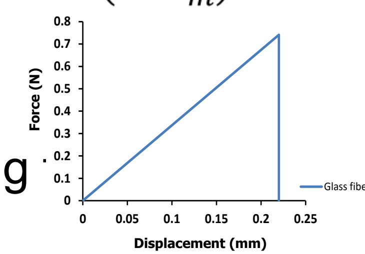
Failure stress for glass-fibers (Weibull Plot)





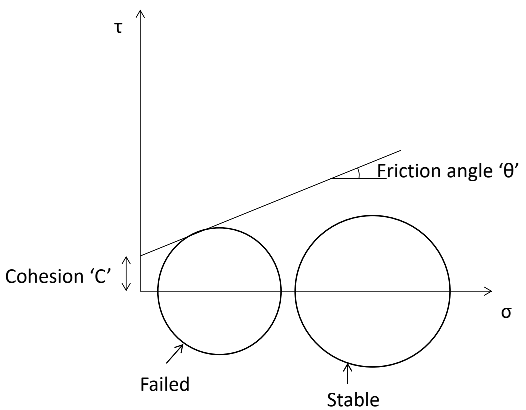
- $P(\sigma) = 1 - e^{\left(-\frac{\sigma}{\sigma_0}\right)^m}$
- $P = \frac{i-0.5}{n}$
- $Y_i = In\left(In\left\{\frac{1}{1-P(\sigma)}\right\}\right)$
- $X_i = In(\sigma_i)$
- $C = -m\, In(\sigma_0)$
- $Y_i = mX_i + C$
- $\sigma_m = \sigma_0 \Gamma\left(1 + \frac{1}{m}\right)$

- Mean value of the tensile strength is 1623.3 MPa
- Maximum principle stress criteria is used for modeling



Matrix failure and debonding

- Yielding takes place when the shear stress, τ, acting on a specific plane reaches a critical value, which is a function of the normal stress, σ_n, acting on that plane.



Elastic definition for traction separation law

$$\begin{Bmatrix} t_n \\ t_s \\ t_t \end{Bmatrix} = \begin{bmatrix} K_{nn} & & \\ & K_{ss} & \\ & & K_{tt} \end{bmatrix} \begin{Bmatrix} \varepsilon_n \\ \varepsilon_s \\ \varepsilon_t \end{Bmatrix}$$

For cohesive layer,

$K_{nn} = K_{ss} = K_{tt} = 35$ Gpa

$t_{nn} = t_{ss} = t_{tt} = 30$ Mpa

Maximum nominal stress criterion

$$\max \left\{ \frac{\langle t_n \rangle}{t_n^0}, \frac{t_s}{t_s^0}, \frac{t}{t} \right.$$

Mohr–Coulomb model for matrix failure

- $\tau = c - \sigma \tan \varphi$

C -cohesion (yield strength of matrix under pure shear loading) = 30 MPa

φ -friction angle (10^0-30^0) = 10^0