Design of Shafts

ME 423: Machine Design
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Introduction

- Torque and Power Transmission
- Most of rotary prime movers either motors or turbines use shaft to transfer the power
- Bearings are required for support
- Shaft failure analysis is critical

Shaft Design

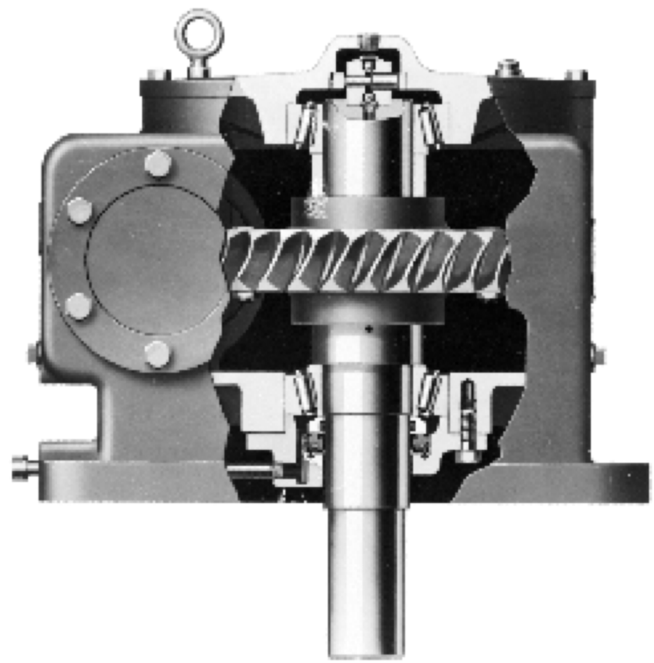
- Material Selection (usually steel, unless you have good reasons)
- Geometric Layout (fit power transmission equipment, gears, pulleys)
- Failure strength
  - Static strength
  - Fatigue strength
- Shaft deflection
  - Bending deflection
  - Torsional deflection
  - Slope at bearings and shaft-supported elements
  - Shear deflection due to transverse loading of short shafts
- Critical speeds at natural frequencies

Shaft Materials

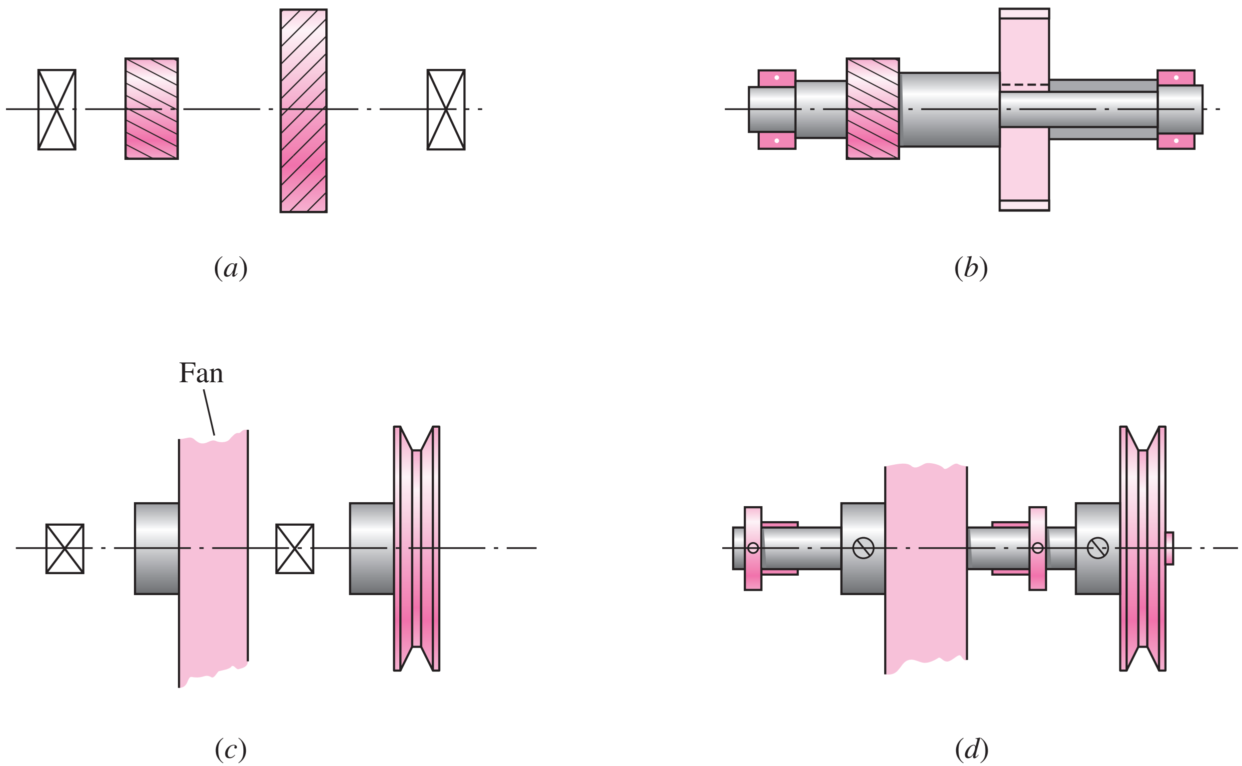
- Deflection primarily controlled by geometry, not material
- Strain controlled by geometry but material has a role in stress
- Strength, Yield or UTS is a material property. Cold drawn steel typical for d< 3 in.
- HR steel common for larger sizes. Should be machined all over.
- Low production quantities: Machining
- High production quantities: Forming

Shaft Layout

- Shafts need to accommodate bearings, gears and pulleys which should be specified
- Shaft Layout
  - Axial layout of components
  - Supporting axial loads (bearings)
  - Providing for torque transmission (gearing/sprockets)
  - Assembly and Disassembly(repair & adjustment)



Axial Layout of Components



(a) (b) (c) (d)

Supporting Axial Load

- Axial loads must be supported through a bearing to the frame
- Generally best for only one bearing to carry axial load to shoulder

Figure 7–3

Tapered roller bearings used in a mowing machine spindle. This design represents good practice for the situation in which one or more torque-transfer elements must be mounted outboard. *(Source: Redrawn from material furnished by The Timken Company.)*

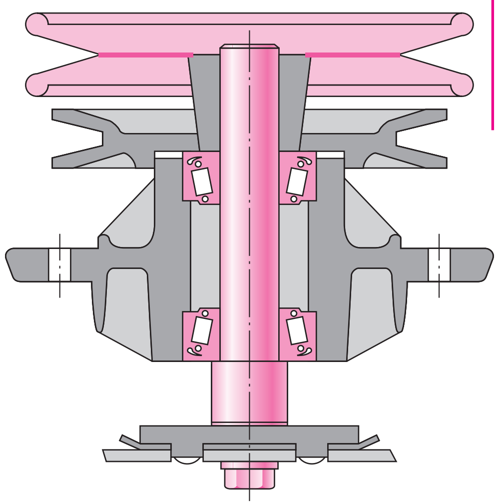
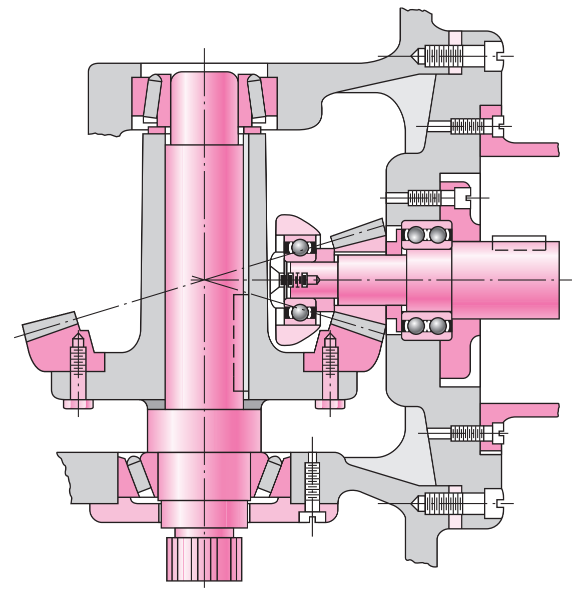


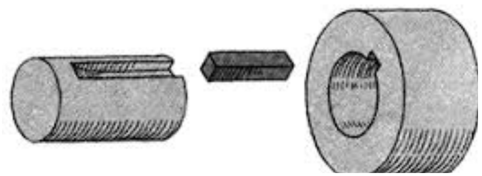
Figure 7–4

A bevel-gear drive in which both pinion and gear are straddle-mounted. *(Source: Redrawn from material furnished by Gleason Machine Division.)*



Torque Transmission

- Common means of transferring torque to shaft
  - Keys
  - Splines
  - Setscrews
  - Pins
  - Press or shrink fits
  - Tapered fits



- Keys are one of the most effective◦
  - Slip fit of component onto shaft for easy assembly
  - Positive angular orientation of component
  - Can design the key to be weakest link to fail in case of overload

Shaft Design for Stresses

- Stresses are only evaluated at critical location
- Critical locations are usually
  - On the outer surface
  - Where the bending moment is large
  - Where the torque is present
  - Where stress concentrations exist

Shaft Stresses

- Standard stress equations can be customized for shafts
- Axial loads are generally small so only bending and torsion will be considered
- Standard alternating and midrange stresses can be calculated

$$\sigma_a = K_f \frac{M_a c}{I} \qquad \sigma_m = K_f \frac{M_m c}{I}$$

$$\tau_a = K_{fs} \frac{T_a c}{J} \qquad \tau_m = K_{fs} \frac{T_m c}{J}$$

$$\sigma_a = K_f \frac{32 M_a}{\pi d^3} \qquad \sigma_m = K_f \frac{32 M_m}{\pi d^3}$$

$$\tau_a = K_{fs} \frac{16 T_a}{\pi d^3} \qquad \tau_m = K_{fs} \frac{16 T_m}{\pi d^3}$$

Design Stresses

- Calculating vonMises Stresses

$$\sigma_a' = (\sigma_a^2 + 3\tau_a^2)^{1/2} = \left[\left(\frac{32K_f M_a}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs} T_a}{\pi d^3}\right)^2\right]^{1/2}$$

$$\sigma_m' = (\sigma_m^2 + 3\tau_m^2)^{1/2} = \left[\left(\frac{32K_f M_m}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs} T_m}{\pi d^3}\right)^2\right]^{1/2}$$

Modified Goodman

- Substituting vonMises into failure criterion

$$\frac{1}{n} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}}$$

$$\frac{1}{n} = \frac{16}{\pi d^3}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs}T_a)^2\right]^{1/2} + \frac{1}{S_{ut}}\left[4(K_f M_m)^2 + 3(K_{fs}T_m)^2\right]^{1/2}\right\}$$

- Solving for diameter

$$d = \left(\frac{16n}{\pi}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs}T_a)^2\right]^{1/2} + \frac{1}{S_{ut}}\left[4(K_f M_m)^2 + 3(K_{fs}T_m)^2\right]^{1/2}\right\}\right)^{1/3}$$

Design of shafts

- Similar approach can be taken with any of the fatigue failure criteria
- Equations are referred to by referencing both the Distortion Energy method of combining stresses and the fatigue failure locus name. For example, DE-Goodman, DE-Gerber, etc.
- In analysis situation, can either use these customized equations for factor of safety, or can use standard approach from Ch. 6.
- In design situation, customized equations for d are much more convenient.

Gerber

- *DE-Gerber*

$$\frac{1}{n} = \frac{8A}{\pi d^3 S_e} \left\{ 1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}} \right)^2 \right]^{1/2} \right\}$$

$$d = \left(\frac{8nA}{\pi S_e} \left\{ 1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}} \right)^2 \right]^{1/2} \right\} \right)^{1/3}$$

where

$$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$$

$$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2}$$

Other Criteria

- ASME Elliptic

$$\frac{1}{n} = \frac{16}{\pi d^3}\left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_a}{S_e}\right)^2 + 4\left(\frac{K_f M_m}{S_y}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2\right]^{1/2} \quad (7\text{–}11)$$

$$d = \left\{\frac{16n}{\pi}\left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_a}{S_e}\right)^2 + 4\left(\frac{K_f M_m}{S_y}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2\right]^{1/2}\right\}^{1/3} \quad (7\text{–}12)$$

- DE Soderberg

$$\frac{1}{n} = \frac{16}{\pi d^3}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{yt}}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\} \quad (7\text{–}13)$$

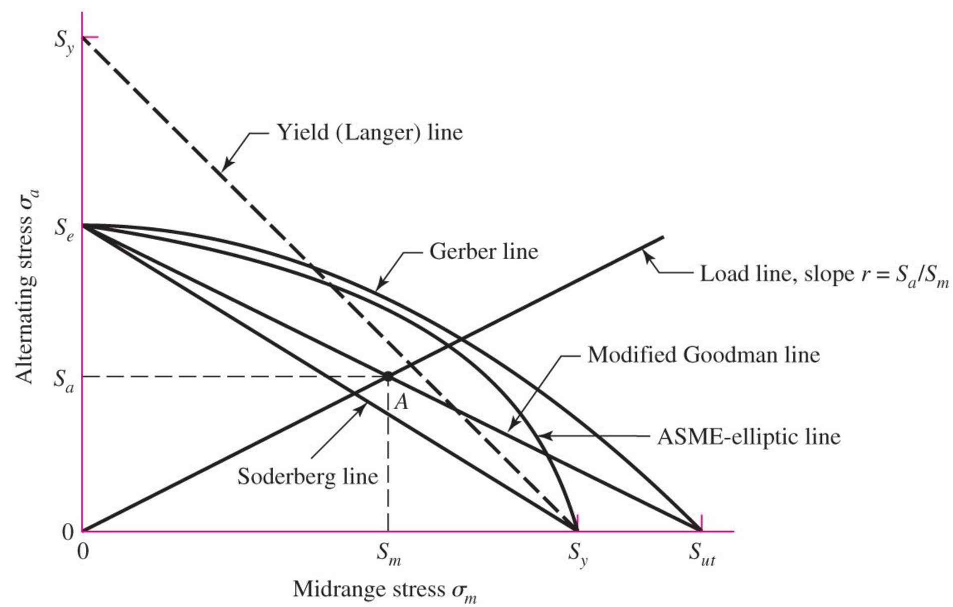
$$d = \left(\frac{16n}{\pi}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{yt}}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\}\right)^{1/3} \quad (7\text{–}14)$$

Rotating Shaft

- For rotating shaft with steady, alternating bending and torsion
  - Bending stress is completely reversed (alternating), since a stress element on the surface cycles from equal tension to compression during each rotation
  - Torsional stress is steady (constant or static)
  - Previous equations simplify with Mm and Ta equal to 0

Yielding Check

- Always necessary to consider static failure, even in fatigue situation
- Soderberg criteria inherently guards against yielding
- ASME-Elliptic criteria takes yielding into account, but is not entirely conservative
- Gerber and modified Goodman criteria require specific check for yielding



Yield Check

- Use von Mises maximum stress to check for yielding,

$$\sigma'_{\max} = \left[(\sigma_m + \sigma_a)^2 + 3(\tau_m + \tau_a)^2\right]^{1/2}$$

$$= \left[\left(\frac{32K_f(M_m + M_a)}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs}(T_m + T_a)}{\pi d^3}\right)^2\right]^{1/2}$$

$$n_y = \frac{S_y}{\sigma'_{\max}}$$

- Alternate simple check is to obtain conservative estimate of σ'max by summing

$$\sigma'_{\max} \approx \sigma'_a + \sigma'_m$$

Deflection Considerations

- Deflection analysis requires complete geometry & loading information for the entire shaft
- Allowable deflections at components will depend on the component manufacturer's specifications.

Table 7–2

Typical Maximum Ranges for Slopes and Transverse Deflections

Slopes	
Tapered roller	0.0005–0.0012 rad
Cylindrical roller	0.0008–0.0012 rad
Deep-groove ball	0.001–0.003 rad
Spherical ball	0.026–0.052 rad
Self-align ball	0.026–0.052 rad
Uncrowned spur gear	< 0.0005 rad
Transverse Deflections	
Spur gears with $P < 10$ teeth/in	0.010 in
Spur gears with $11 < P < 19$	0.005 in
Spur gears with $20 < P < 50$	0.003 in

Determination of deflections

- Linear & angular deflections, should be checked at gears and bearings
- Deflection analysis is straightforward, but very lengthy and tedious to carry out manually. Consequently, shaft deflection analysis is almost always done with the assistance of software(usually FEA)
- For this reason, a common approach is to size critical locations for stress, then fill in reasonable size estimates for other locations, then check deflection using FEA or other software
- Software options include specialized shaft software, general beam deflection software, and finite element analysis (FEA) software.

Critical Speeds

- For a rotating shaft if the centripetal force is equal to the elastic restoring force, the deflection increases greatly and the shaft is said to "whirl"
- Below and above this speed this effect is not pronounced
- This critical (whirling speed) is dependent on:
  - The shaft dimensions
  - The shaft material and
  - The shaft loads

Critical speeds of shafts

Force balance of restoring force and centripetal,

$$m\omega^2 y = ky$$

k is the stiffness of the transverse vibration

$$\omega = 2\,\pi N_c = \sqrt{\frac{k}{m}}$$

The critical speed for a point mass of m,

$$N_c = \frac{1}{2\,\pi}\sqrt{\frac{k}{m}}$$

For a horizontal shaft,

$$N_c = \frac{1}{2\,\pi}\sqrt{\frac{g}{y}}$$

Where y = the static deflection at the location of the concentrated mass

Ensemble of lumped masses

- For ensemble of lumped masses Raleigh's method pf lumped masses gives,

$$\omega_1 = \sqrt{\frac{g \sum w_i y_i}{\sum w_i y_i^2}}$$

- where w_i is the weight of the i^{th} location and y_i is the deflection at the i^{th} body location

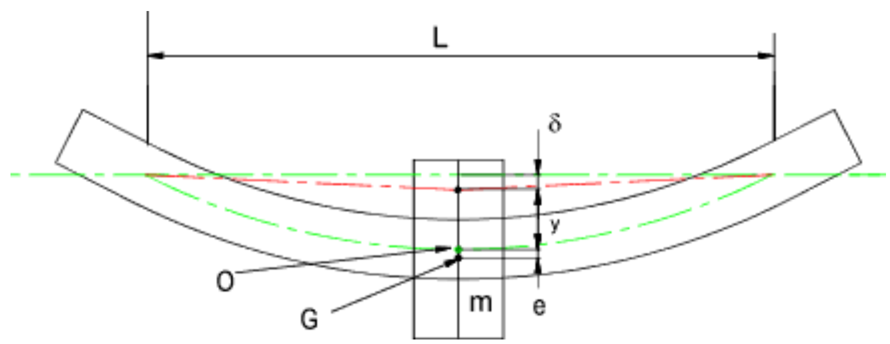
Beam Theory

- m = Mass (kg)
- Nc = critical speed (rev/s)
- g = acceleration due to gravity (m.s-2)
- O = centroid location
- G = Centre of Gravity location
- L = Length of shaft
- E = Young's Modulus (N/m2)
- I = Second Moment of Area (m4)
- y = deflection from δ with shaft rotation = ω δ static deflection (m)
- ω = angular velocity of shaft (rads/s)

Whirling Speed

- The centrifugal force on the shaft = $m\,\omega^2(y + e)$ and the inward pull exerted by the shaft, $F = y48EI / L^3$ for simply supported. For a general beam $F = y\,K\,EI / L^3$

where K is constant depending on the loading and the end support conditions

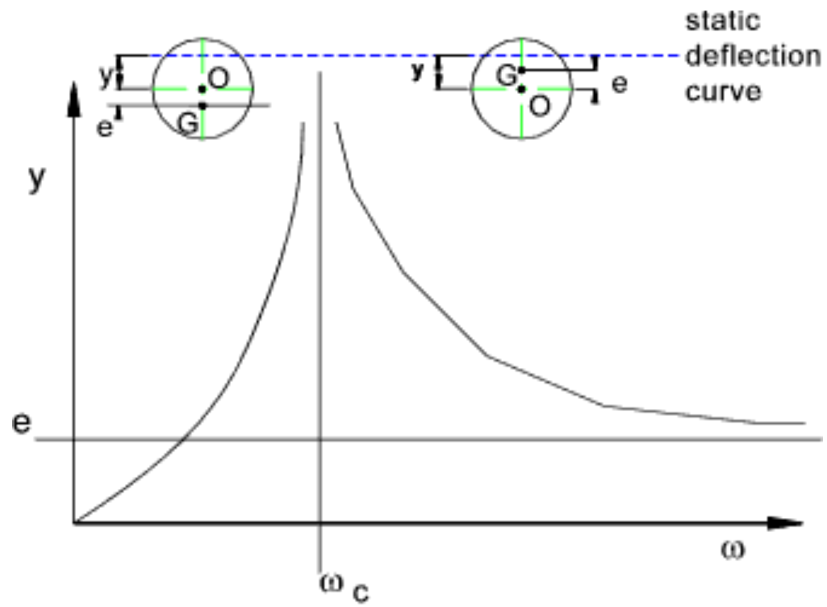


$$m\,\omega^2(y + e) = y\frac{KEI}{L^3} \quad \text{therefore} \quad y = \frac{e}{\left(\frac{KEI}{m\omega^2 L^3} - 1\right)}$$

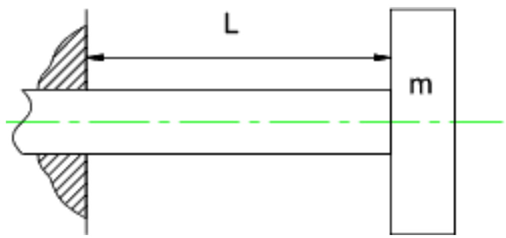
Critical Speed

- The critical speed is given by

$$y = \frac{\omega^2 . e}{(\omega_c^2 - \omega^2)}$$

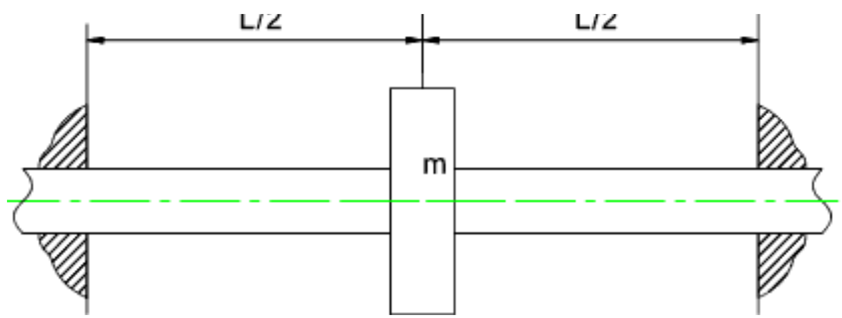


Critical speeds of some configurations



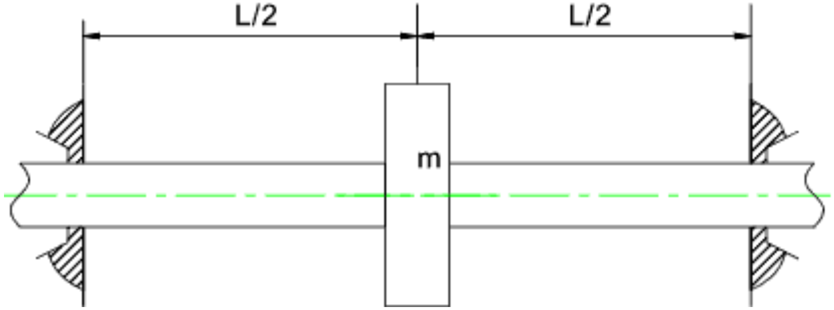
$$N_c = \frac{\sqrt{3EI\ /\ mL^3}}{2\pi}$$

Cantilevered Shaft
with disc at end



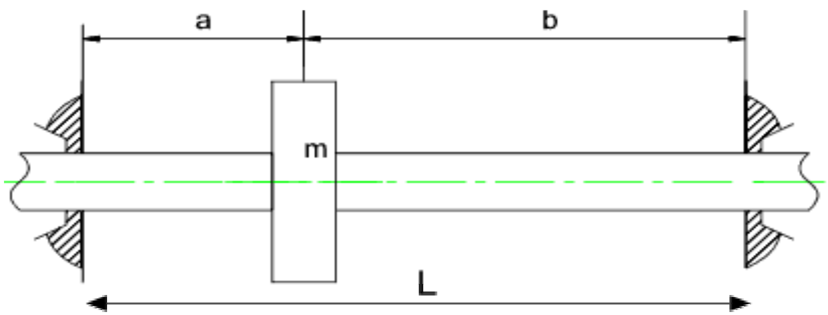
$$N_c = \frac{\sqrt{192EI\ /\ mL^3}}{2\pi}$$

Central Disc
with long bearings



$$N_c = \frac{\sqrt{48EI \;/\; mL^3}}{2\,\pi}$$

Central Disc
with short bearings



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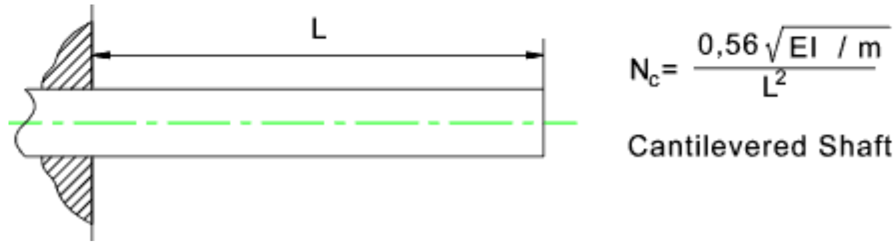
$$N_c = \frac{\sqrt{3EIL\;/ma^2\,b^2}}{2\,\pi}$$

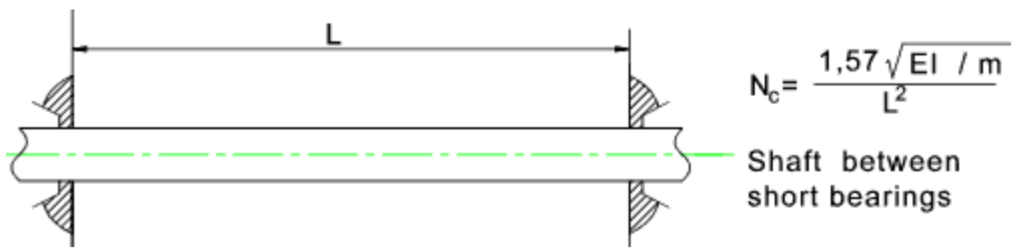
Non-central disc
with short bearings

Cantilevered Shaft

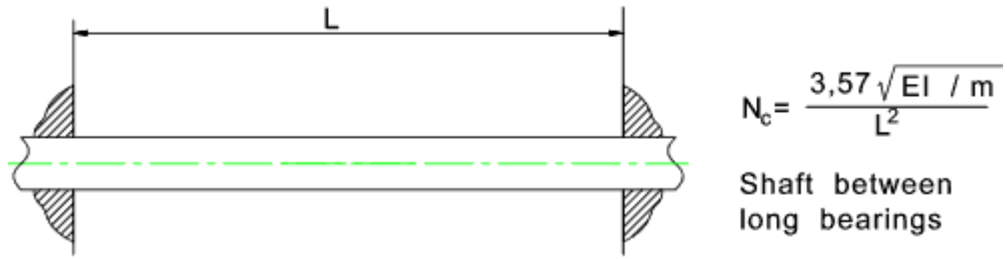


m = mass /unit length

Shaft Between short bearings



m = mass /unit length



m = mass /unit length

Dunkerley's Method

This is known as Dunkerley's method an is based on the theory of superposition....

$$\frac{1}{N_c^2} = \frac{1}{N_s^2} + \frac{1}{N_1^2} + \frac{1}{N_2^2} + \ldots\ldots$$