Outline

- Principal stresses
- Mohr's circle in 3D
- Strain tensor
- Principal strains

Principal Stresses in 3D

- 3-D Stresses can be represented by in usual notation

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

We will use a concept from continuum mechanics

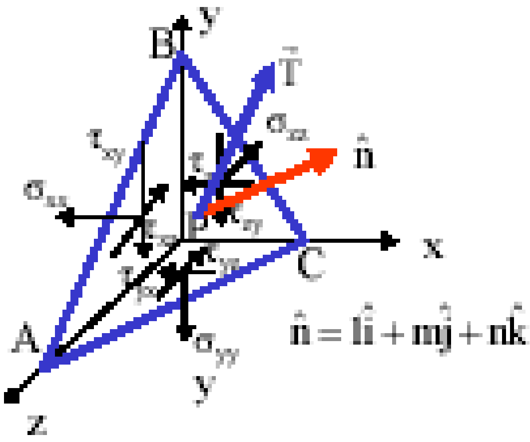
$$\sigma \cdot \hat{n} = \vec{T}$$

Traction vector
Force/area

Stress Tensor

Unit normal vector

Principal Stresses



Principal Stresses in 3D

$$\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \sigma \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

Principal Stresses in 3D

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

3D Stress – Principal Stresses

The three principal stresses are obtained as the three real roots of the following equation:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

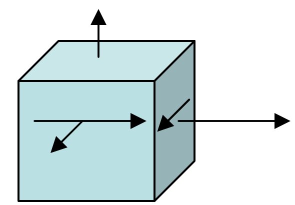
$$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2$$

I_1, I_2, and I_3 are known as **stress invariants** as they do not change in value when the axes are rotated to new positions.

Principal Stress

$$\begin{bmatrix} 0 & -240 & 0 \\ -240 & 200 & 0 \\ 0 & 0 & -280 \end{bmatrix}$$



```
In[3]:= Det[{{0 - σ, -240, 0}, {-240, 200 - σ, 0}, {0, 0, -280 - σ}}]
```

Out[3]= $16\,128\,000 + 113\,600\,\sigma - 80\,\sigma^2 - \sigma^3$

```
In[2]:= Solve[Det[{{0 - σ, -240, 0}, {-240, 200 - σ, 0}, {0, 0, -280 - σ}}] == 0,
  σ]
```

Out[2]= {{σ → -280}, {σ → -160}, {σ → 360}}

```
In[1]:= Eigenvalues[{{0, -240, 0}, {-240, 200, 0}, {0, 0, -280}}]
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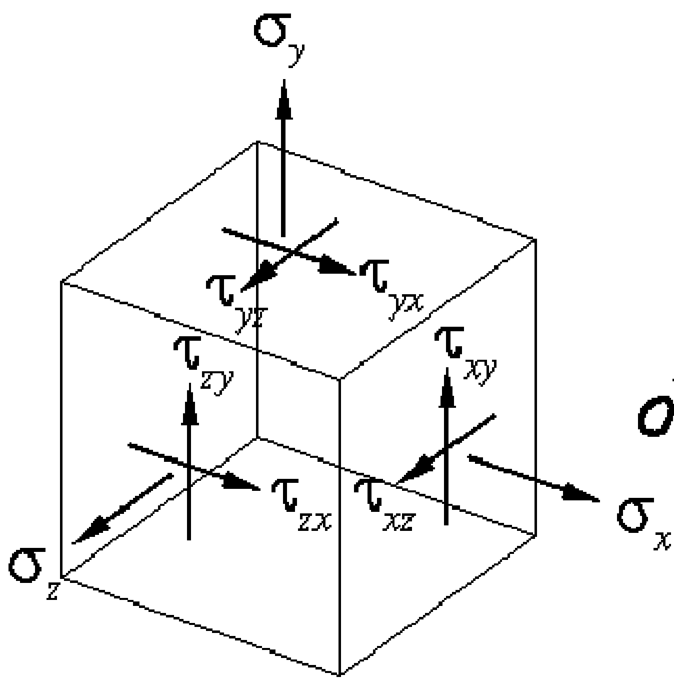
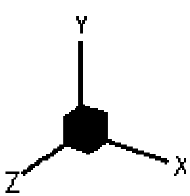
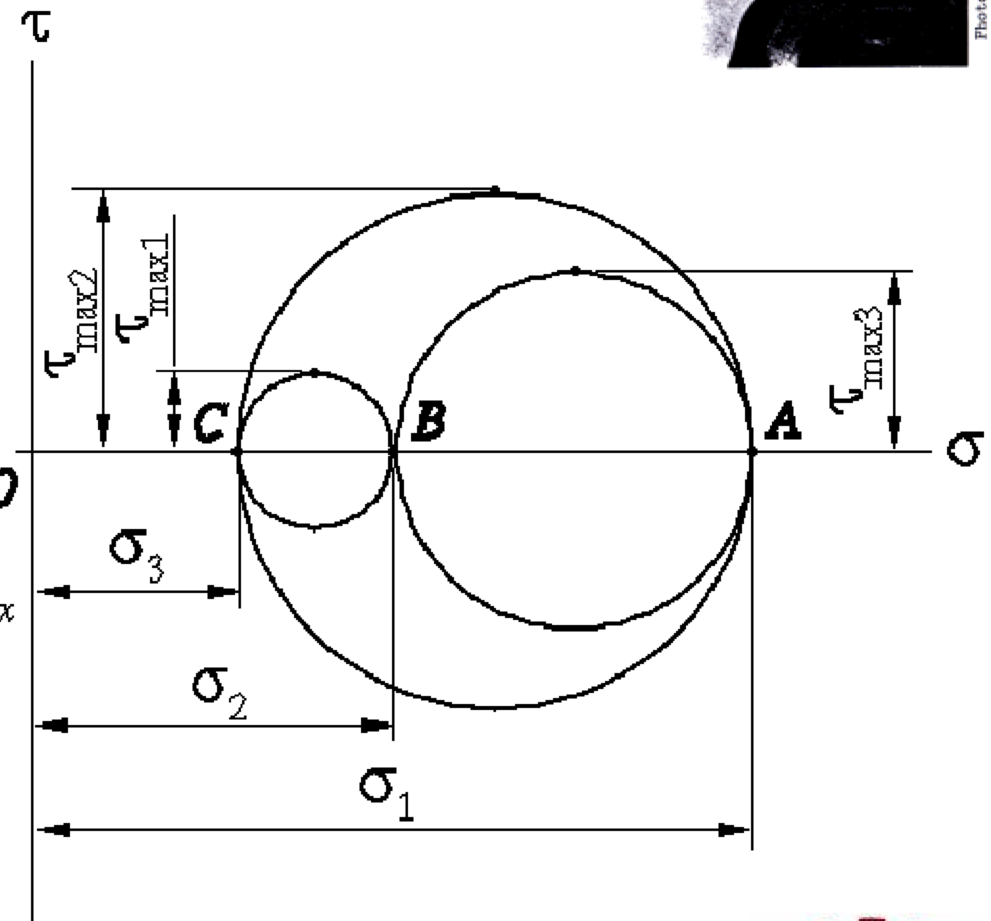
Out[1]= {360, -280, -160}

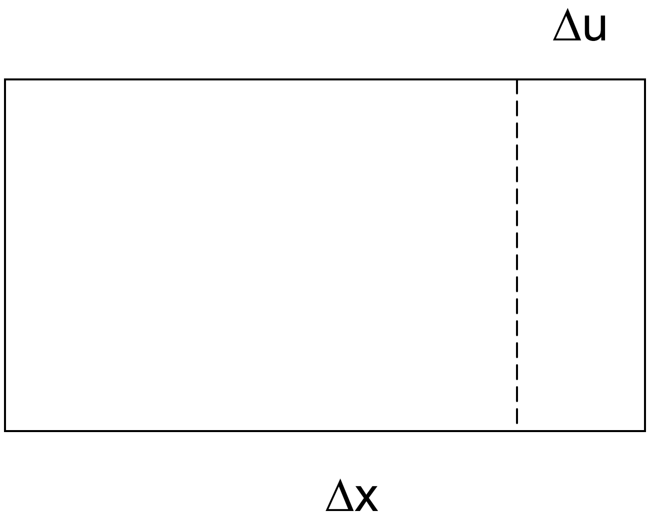
Principal Stresses in 3-D







Linear Strains



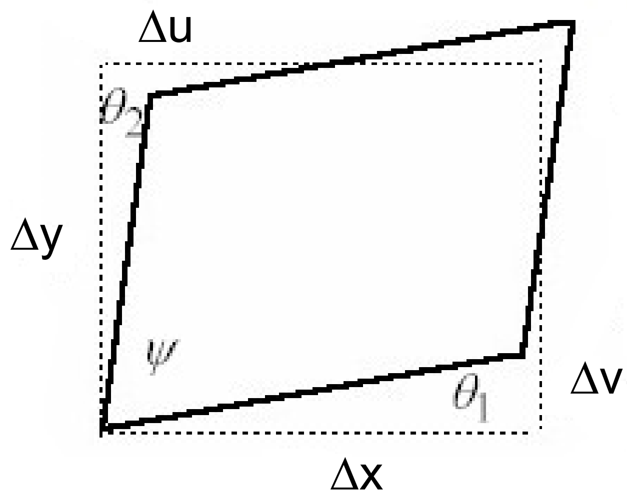
Linear strain formulation:

$$\varepsilon_x = \frac{\Delta u}{\Delta x}$$

Taking limits it can be represented as,

$$\varepsilon_x = \frac{\partial u}{\partial x}; \varepsilon_y = \frac{\partial v}{\partial y}; \varepsilon_z = \frac{\partial w}{\partial z}$$

Shear Strain



$$\gamma_{xy} = \frac{\pi}{2} - \psi = \theta_1 + \theta_2$$

$$\tan\theta_1 \approx \theta_1 \approx \frac{\Delta v}{\Delta x}$$

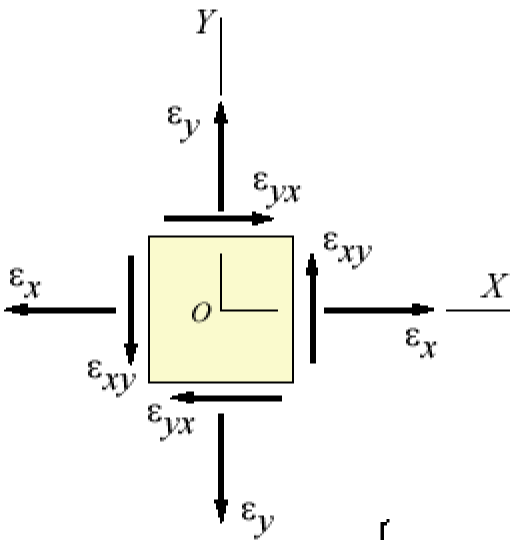
$$\tan\theta_2 \approx \theta_2 \approx \frac{\Delta u}{\Delta y}$$

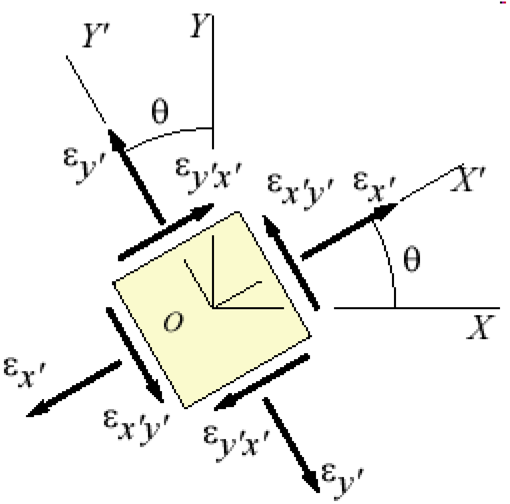
$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}(\theta_1 + \theta_2) = \frac{1}{2}\left(\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Strain Tensor

$$\varepsilon_{i,j} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) & \dfrac{\partial w}{\partial z} \end{bmatrix}$$

Strain Transformation





$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$$

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \varepsilon_{xy}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \varepsilon_{xy}\sin 2\theta$$

$$= \varepsilon_x + \varepsilon_y - \varepsilon_{x'}$$

$$\varepsilon_{x'y'} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \varepsilon_{xy}\cos 2\theta$$

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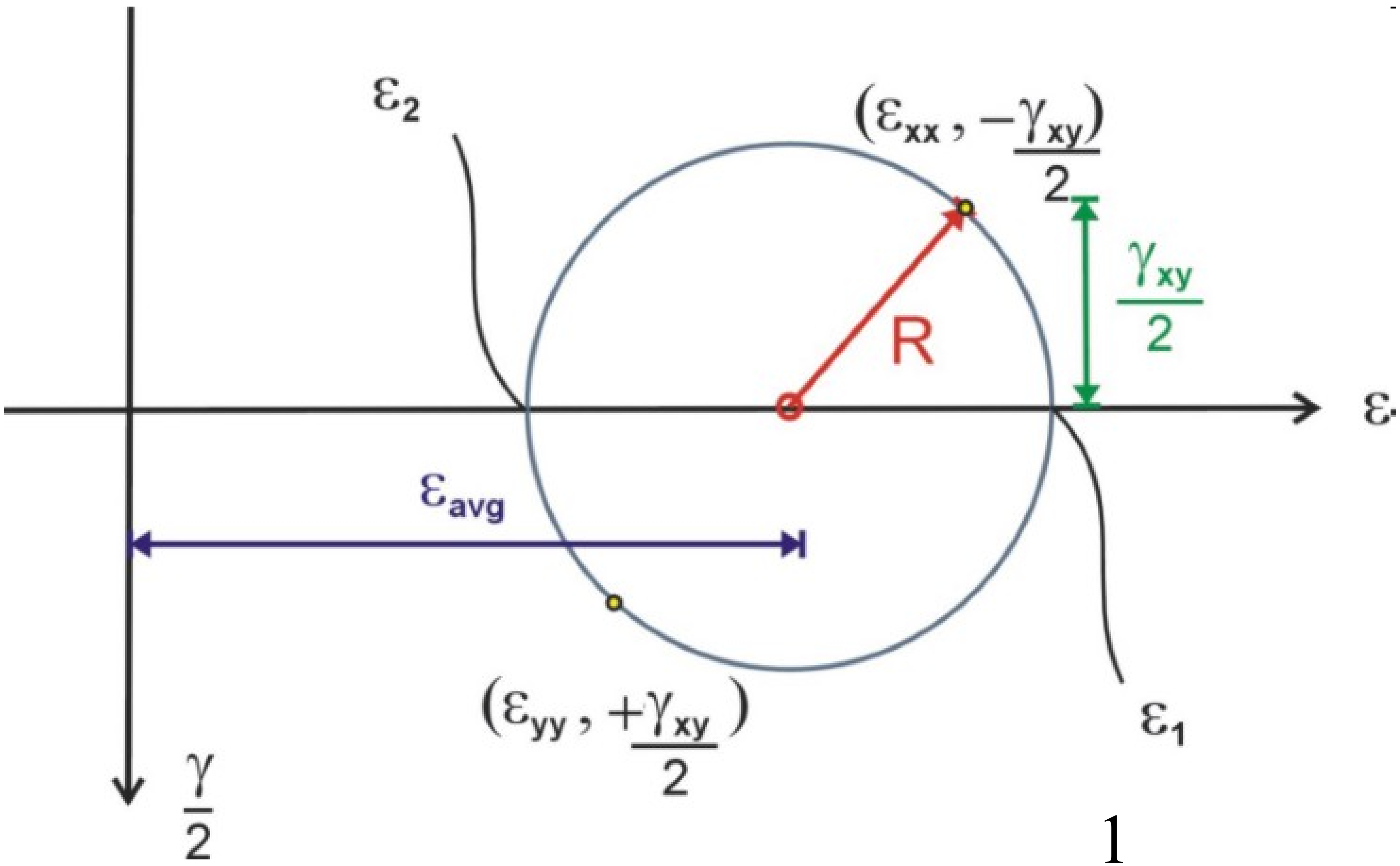
Mohr's Circle for Strain



$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$$

Principal Strains

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2}$$

where,

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$$