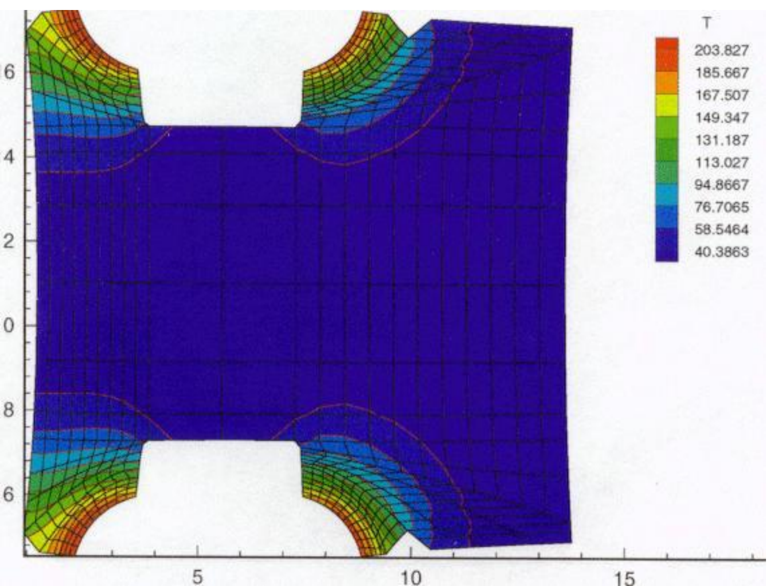
Analytical Modeling of Laser Moving Sources



Contains:

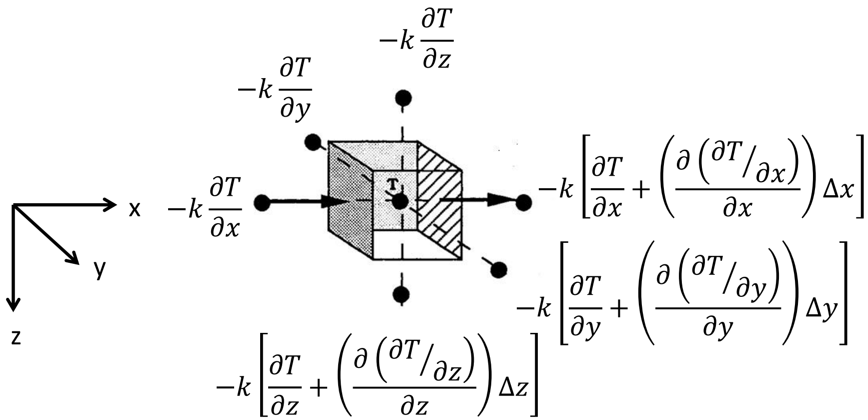
- Heat flow equation
- Analytic model in one dimensional heat flow
- Heat source modeling
  - Point heat source
  - Line heat source
  - Plane heat source
  - Surface heat source
- Finite difference formulation
- Finite elements

Heat flow equation

For developing basic heat flow equation,

consider the differential element .

Heat balance in element is given by,



Heat flow through differential element

Heat in – Heat out + Heat generated

= Heat accumulated

Heat in and out rates depends on conduction and convection.

For x axis,

Heat in- Heat (conduction)

$$-k\frac{\partial T}{\partial x}-\left(-k\left[\frac{\partial T}{\partial x}+\left(\frac{\partial\left(\partial T/\partial x\right)}{\partial x}\right)\Delta x\right]\right)\Delta y\Delta z$$

Heat in- Heat (convection or advection)

$$\rho C_p T U_x \Delta y\Delta z-\rho C_p U_x\left(T+\frac{\partial T}{\partial x}\Delta x\right)\Delta y\Delta z$$

Heat Flow Equation

$$heat\ accumulation = \rho C_p \frac{\partial T}{\partial t} \Delta x \Delta y \Delta z$$

$$heat\ generated = H \Delta x \Delta y \Delta z$$

$$k\nabla^2 T \Delta x \Delta y \Delta z - \rho C_p U \nabla T \Delta x \Delta y \Delta z + H \Delta x \Delta y \Delta z = \rho C_p \frac{\partial T}{\partial t} \Delta x \Delta y \Delta z$$

$$k\nabla^2 T - \rho C_p \frac{\partial T}{\partial t} - \rho C_p U \nabla T = -H$$

One dimensional heat conduction

- If the heat flow in only one direction and there is no convection or heat generation, the basic equation becomes

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ where α = diffusivity, t = time

- Using separation of variables, the solution can be assumed to be a product of spatial variable, u(z), and time variable, v(t):

$$T(z,t) = u(z)\, v(t)$$

$$v(t)\ \frac{\partial^2 u(z)}{\partial z^2} = \frac{u(z)}{\alpha}\frac{\partial v(t)}{\partial t}$$

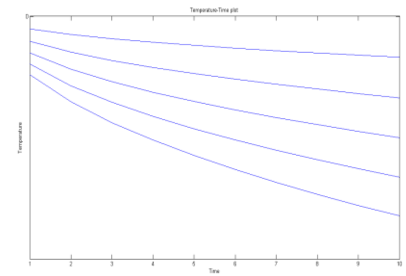
$$\frac{1}{u(z)}\ \frac{\partial^2 u(z)}{\partial z^2} = \frac{1}{\alpha\, v(t)}\frac{\partial v(t)}{\partial t} = -\beta^2$$

$$u(z) = A\cos(\beta z) + B\sin(\beta z)$$
$$v(t) = Ce^{-\alpha\beta^2 t}$$

Flux formulation in one dimension

If the heat flow in only one direction with a flux input at z=0

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (1)$$



$$q(z,t) = -k\frac{\partial T(z,t)}{\partial z} \qquad (2)$$

Differentiating Eq. (2) with respect to space variable

$$\frac{\partial^2 q}{\partial z^2} = -k\frac{\partial^3 T}{\partial z^3} \qquad (3)$$

The one dimensional heat conduction equation given by Eq. (1) is differentiated with space variable,

$$\frac{\partial^3 T}{\partial z^3} = \frac{1}{\alpha}\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial t}\right) \qquad (4)$$

Differentiating Eq. (2) with respect to time variable yields,

$$\frac{\partial q}{\partial t} = -k\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial z}\right) \qquad (5)$$

Flux formulation

By manipulating Eq. (3) to (5)

$$\frac{\partial^2 q}{\partial z^2} = \frac{1}{\alpha}\left(\frac{\partial q}{\partial t}\right) \quad \text{in } 0 < z < \infty, t > 0 \qquad (6)$$

The boundary and the initial conditions are given by,

$q(z,t) = f_0$ at $z = 0, t > 0$

$q(z,t) = 0$ at $t = 0$

The solution for this is given by,

$$T(z,t) = \frac{2\,f_0}{k}\left[\left(\frac{\alpha\,t}{\pi}\right)^{\frac{1}{2}} e^{-\frac{z^2}{4\alpha t}} - \frac{z}{2}\operatorname{erfc}\left(\frac{z}{\sqrt{4\alpha t}}\right)\right]$$

At the surface (z=0) the temperature is

$$T(0,t) = \frac{2\,f_0}{k}\left(\frac{\alpha\,t}{\pi}\right)^{\frac{1}{2}}$$

Heat source modelling: Introduction:

- Why modeling?

1. Semi-quantitative understanding of the process mechanism for the design of experiments.

2. Parametric understanding for control purpose. E.g. statistical charts.

3. Detailed understanding to analyse the precise process mechanisms for the purpose of prediction, process improvement .

Types of heat sources:

Point heat source.

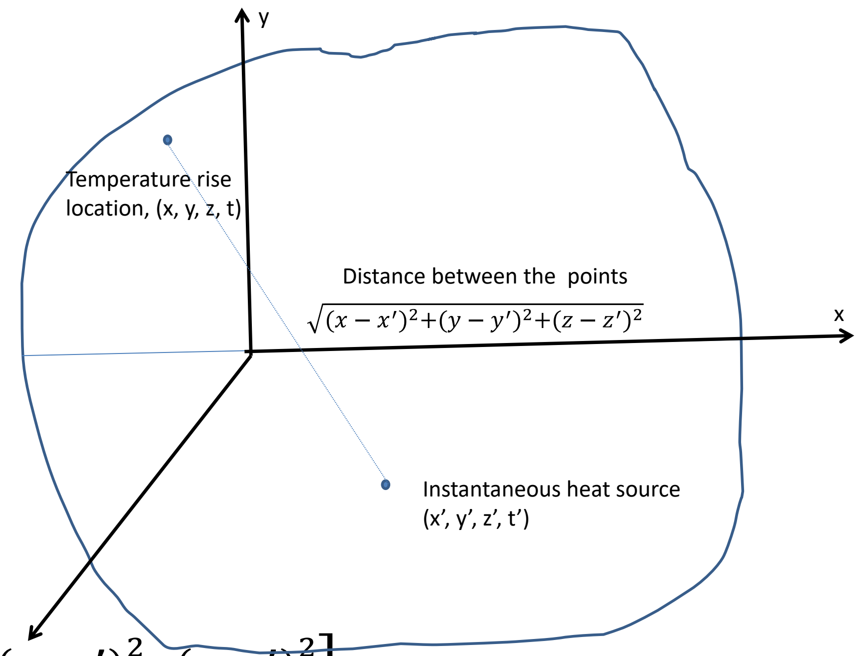
Line heat source.

Plane heat source. (e.g. circular , rectangular)

1.Instantaneous point heat source:

The differential equation for the conduction of heat in a stationary medium assuming no convection or radiation, is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + q = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$



This is satisfied by the

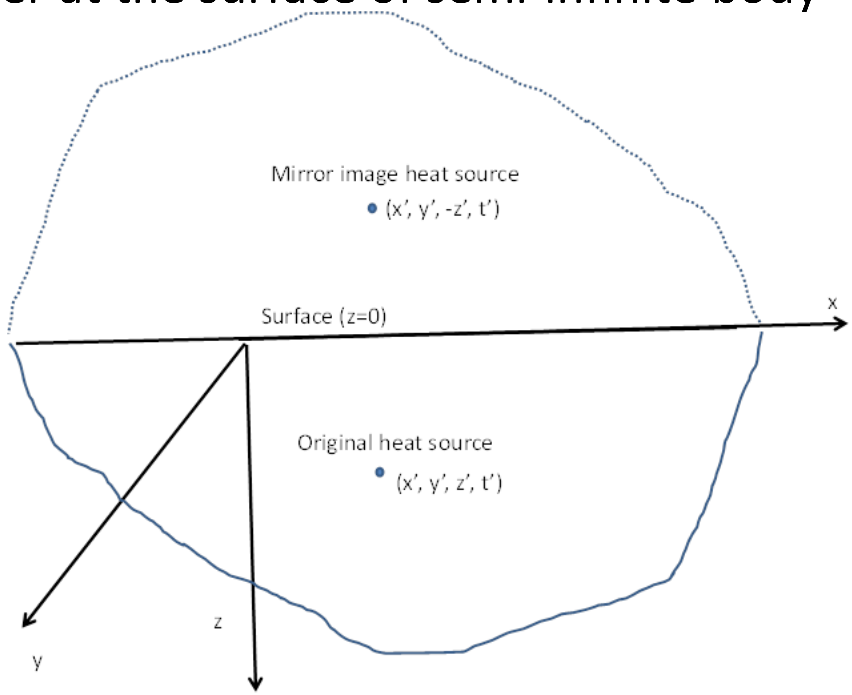
solution for infinite body,

$$\mathrm{d}T(x,y,z,t) = \frac{\delta q}{\rho C\left(4\pi a(t-t')\right)^{3/2}} exp\left[-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4a(t-t')}\right]$$

gives the temperature rise at position (x, y, z) and time t due to an instantaneous heat source δq applied at position (x', y', z') and time t'; where δq = instantaneous heat generated, C = sp. heat capacity, α = diffusivity, ρ = Density, t = time, K = thermal conductivity.

Temperature rise in semi-infinite body

- A mirror image can be used
- No heat transfer at the surface of semi-infinite body



$$dT(x,y,z,t) = \frac{\delta q}{\rho C\left(4\pi a(t-t')\right)^{3/2}} exp\left[-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4a(t-t')}\right]$$
$$+\frac{\delta q}{\rho C\left(4\pi a(t-t')\right)^{3/2}} exp\left[-\frac{(x-x')^2+(y-y')^2+(z+z')^2}{4a(t-t')}\right]$$

Semi-infinite body

- $dT(x,y,z,t) = \frac{\delta q}{\rho C\left(4\pi a(t-t')\right)^{3/2}} exp\left[-\frac{(x-x')^2+(y-y')^2}{4a(t-t')}\right]\left[exp\left[-\frac{(z-z')^2}{4a(t-t')}\right] + exp\left[-\frac{(z+z')^2}{4a(t-t')}\right]\right]$
- If the heat is applied at the surface or z'=0, such as moving area heat sources
- $dT(x,y,z,t) = \frac{2\delta q}{\rho C\left(4\pi a(t-t')\right)^{3/2}} exp\left[-\frac{(x-x')^2+(y-y')^2+z^2}{4a(t-t')}\right]$

Results:

1. When (x, y, z) = (0,0,0) and (x', y', z') = (0,0,0)

   T = 473.3379

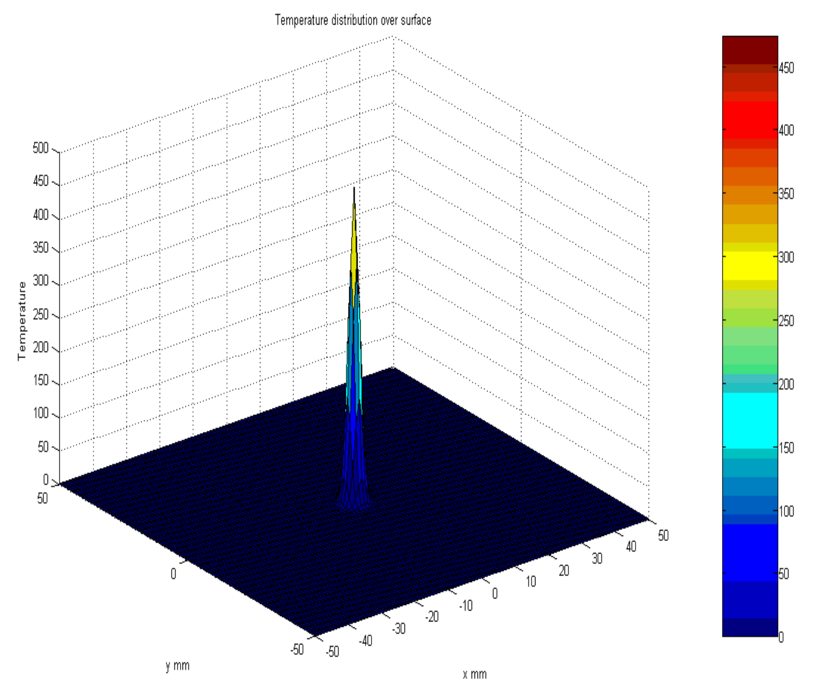
2. When (x, y, z) = (0.5,0.5,0) and (x', y', z') = (0,0,0)

   T = 413.0811

3. When (x, y, z) = (2,2,0) and (x', y', z') = (0,0,0)

   T = 53.5792

For temperature over entire surface consider heat source at (0,0,0) and workpiece have dimension 50 X 50. Temperature distribution Is shown in figure.



2. Continuous point heat source in infinite body:

If the heat is liberated at the rate dQ= P.dt’ from t = t’ to t = t’+ dt’ at the point (x’, y’, z’), the temperature at (x, y, z) at time t is found by integrating above equation, and C = sp. heat capacity, α = diffusivity, ρ = Density. From the point heat source solution,

$$dT\left(x,y,z,t\right)=\frac{Pdt'}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\exp[-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4a(t-t')}]$$

Now if the heat source is on from time t’=0 to t’=t continuously it can be written as

$$dT(x,y,z,t)=\int_{t'=0}^{t'=t}\frac{Pdt'}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\exp\left[-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4a(t-t')}\right]$$

Continuous point heat source

To simplify the situation, one can assume that the heat source was switched on at time, t'=-t and turned off at t'=0

$$dT(x,y,z,t) = \int_0^t \frac{P dt}{\rho C (4\pi a(t))^{\frac{3}{2}}} \exp\left[-\frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{4a(t)}\right]$$

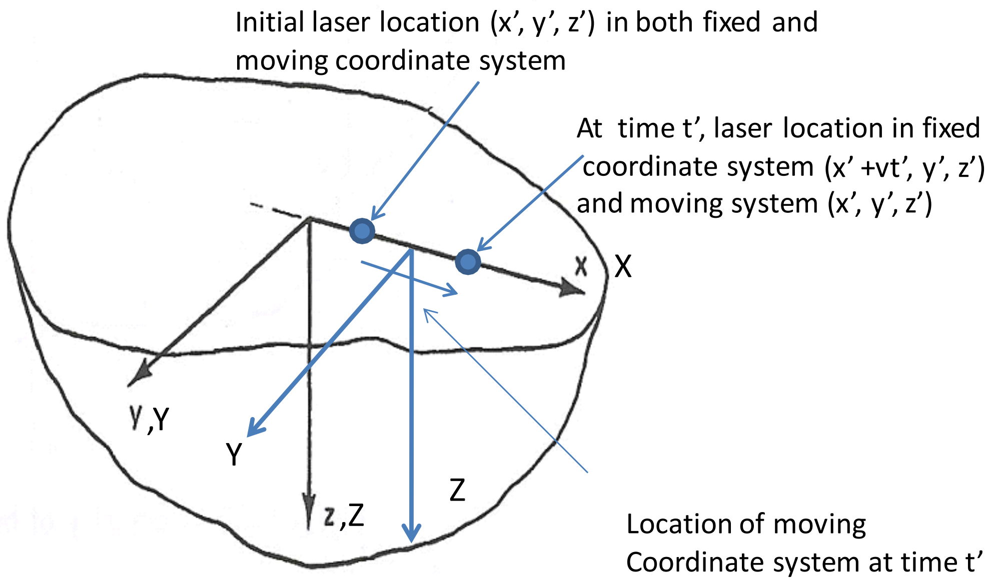
where Q is in Watts. As $t \rightarrow \infty$ steady state temperature distribution occurs given by

$$T(x,y,z) = \frac{P}{4\,\pi k} \exp[-(x-x')^2 + (y-y')^2 + (z-z')^2]$$

Moving heat source solution steps

- Moving heat source is in fact a continuous stationary source in moving frame of reference
- Next step is used to find the superposition of point solutions in spatial co-ordinates in moving frame of reference for obtaining, line, plane or volumetric heat source.
- Transform the solution to fixed coordinate system
- Integrate with respect to time (t') for final solution in T(x, y, z, t)

Moving point heat source in semi-infinite body



Moving laser source along X -axis in a semi -infinite body

In moving coordinate system:

$$dT(X,Y,Z,t) = \frac{2\delta q}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}} \exp[-\frac{(X-x')^2+(Y-y')^2+(Z)^2}{4a(t-t')}]$$

In fixed coordinate system:

$$dT(x,y,z,t) = \frac{2\delta q}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}} \exp[-\frac{(x-vt'-x')^2+(y-y')^2+(z)^2}{4a(t-t')}]$$

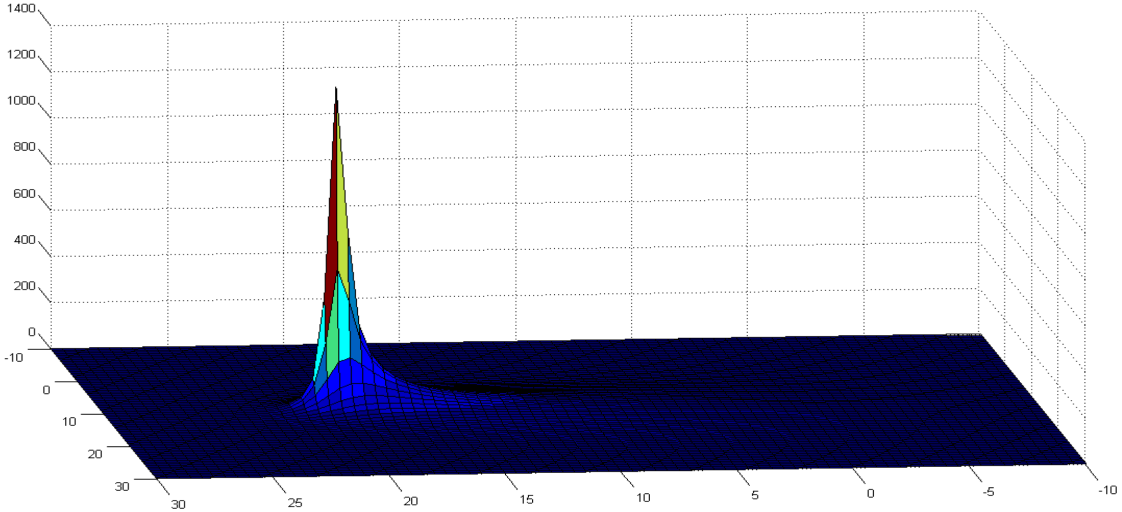
Note that $\delta q = Pdt'$

Moving point heat source:

Consider point heat source P heat units per unit time moving with velocity v on semi-infinite body from time t'= 0 to t'= t. During a very short time heat released at the surface is dQ = Pdt'. This will result in infinitesimal rise in temperature at point (x, y, z) at time t given by,

$$dT\left(x,y,z,t\right)=\int_{t'=0}^{t'=t}\frac{2Pdt'}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\exp[-\frac{(x-vt'-x')^2+(y-y')^2+(z-z')^2}{4a(t-t')}]$$

The total rise in of the temperature can be obtained by
integrating from t'=0 to t'= t



Line heat source in infinite body:

Temperature for the line heat source can be obtained directly by integrating the solution of the point source in the moving coordinate system.

- line source in moving coordinate:

Line source parallel to z-axis and passing through point (x' , y') in moving system. The temperature obtained by integrating , where C = sp. heat capacity, ρ = Density, K = thermal conductivity. Here Q_l = heat per unit length

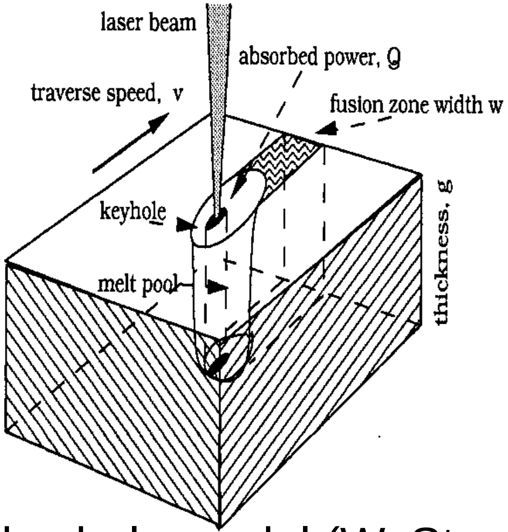


Fig. keyhole model (W. Steen)

For infinite body

$$dT'(X,Y,Z,t)=\frac{\delta q}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\exp[-\frac{(X-x')^2+(Y-y')^2+(Z-z')^2}{4a(t-t')}]$$

This point source in moving coordinates can be superposed for infinite line along z,

$$dT'(X,Y,t)=\frac{q_l dt'}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\int_{-\infty}^{\infty}\exp[-\frac{(X-x')^2+(Y-y')^2+(Z-z')^2}{4a(t-t')}]dz$$

Infinite line source

- Integrating in moving coordinate system with respect to spatial variables,

$$dT(\mathrm{X}, \mathrm{Y}, t) = \frac{q_l dt'}{4\pi k(t-t')} \exp\left[-\frac{(X-x')^2 + (Y-y')^2}{4a(t-t')}\right]$$

- Convert to stationary frame and integrate to time

X= x-vt', Y=y and Z=z

$$T - T_0(x, y, t) = \int_{t'=0}^{t} \frac{q_l\, dt'}{4\pi k(t-t')} \exp\left[-\frac{(\mathrm{x} - \mathrm{vt}' - x')^2 + (\mathrm{y} - y')^2}{4a(t-t')}\right]$$

This can be integrated numerically

Moving line heat source

- Using the same concept used in stationary continuous point where the laser (heat source) started at t' =-τ, and at time τ=0 the laser source is at origin (x'=0 and y'=0). One can get solution at (X, Y) from laser source:

$$T(\mathrm{X},\mathrm{Y},t) = \int_0^t \frac{q_l\, d\tau}{4\pi k(\tau)} \exp\left[-\frac{(\mathrm{X}+\mathrm{v}\tau)^2 + Y^2}{4a(\tau)}\right]$$

Similar result can be obtained by transformation,

$$t - t' = \tau$$

$$x - vt' = x + v(\tau - t) = x - vt + v\tau$$

At time t, x-vt =X, location in moving or laser coordinate system

$x - vt' = x - vt + v\tau = X + v\tau$ and $d\tau = -dt'$

The limits, at $t' = 0, \tau = t\ and t' = t, \tau = 0$

$$T(\mathrm{X},\mathrm{Y},t) = \int_t^0 -\frac{q_l\, d\tau}{4\pi k(\tau)} \exp\left[-\frac{(\mathrm{X}+\mathrm{v}\tau)^2 + Y^2}{4a(\tau)}\right]$$

This also needs to be integrated numerically

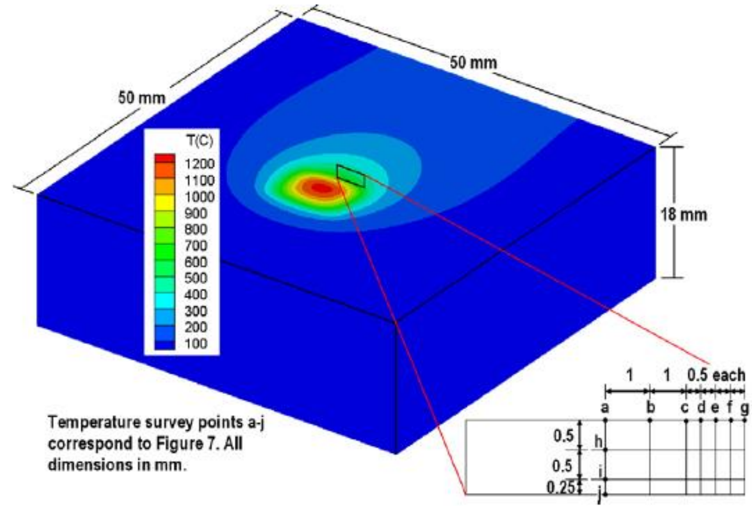
The steady state solution at $t \to \infty$,

$$T(x, y) = \frac{q_l}{2\pi k} e^{-\frac{v\,X}{2\alpha}} BesselK\left(0, \frac{v\sqrt{X^2 + Y^2}}{2\alpha}\right)$$

Bessel function of second kind 0 order

It may be noted that it is a steady-state solution and X, Yare from the laser center.

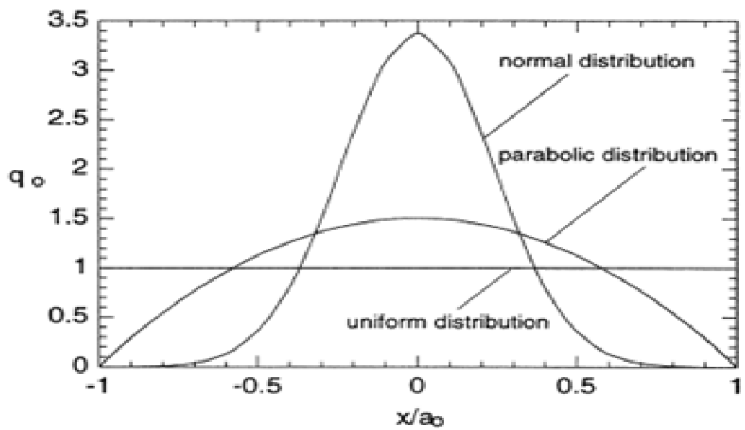
Plane heat source:



Surface heat source:

- Area (circular, rectangular heat source)
- Applied on x-y plane.
- Temperature depends on intensity.
- Application: surface hardening,

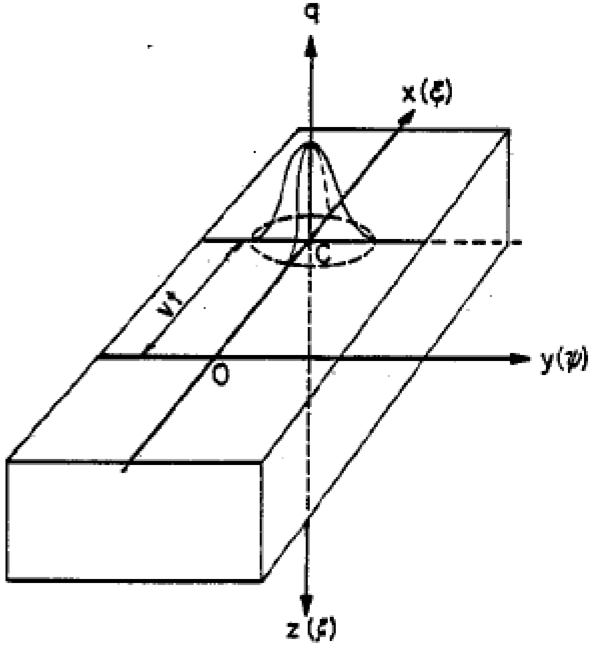
surface cladding etc.



Gaussian moving circular heat source:

Gaussian heat source intensity

$$I(x,y) = \frac{2*P}{\pi\sigma^2}\text{Exp}[-\frac{2(x^2+y^2)}{\sigma^2}]$$



In moving coordinate system,

$$dT(X,Y,Z,t') = \frac{2\delta q}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\exp[-\frac{(X-x')^2+(Y-y')^2+(Z)^2}{4a(t-t')}]$$

$$dT = \frac{2dt'}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}} I(x',y')dx'dy'\exp[-\frac{(X-x')^2+(Y-y')^2+(Z)^2}{4a(t-t')}]$$

Superposing the point solutions for the Gaussian beam,

$$dT(t') = \frac{4Pdt'}{\pi\sigma^2\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\times$$

$$\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \exp[-(\frac{2x'^2+2y'^2}{\sigma^2}+\frac{x'^2-2(X)x'+(X)^2+y'^2-2Yy'+Y^2+Z^2}{4a(t-t')})]$$

Moving heat source.

Where P = laser power, σ = beam radius, v = scanning velocity, a = diffusivity, t =time.

Rewriting the solution for fixed coordinate system,

$$dT(t) = \frac{4Pdt'}{\pi\sigma^2\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\frac{\pi\sigma^2 4a(t-t')}{\sigma^2+8a(t-t')}\exp[-\frac{2((x-vt')^2+y^2)}{\sigma^2+8a(t-t')}-\frac{z^2}{4a(t-t')}]$$

$$T-T_0 = \frac{4P}{\rho C\pi\sqrt{4a\pi}}\int_{t'=0}^{t'=t}\frac{dt'(t-t')^{-0.5}}{\sigma^2+8a(t-t')}\exp[-\frac{2((x-vt')^2+y^2)}{\sigma^2+8a(t-t')}-\frac{z^2}{4a(t-t')}]$$

Numerical integration can be carried out for the above equation

If solution is required in moving or laser coordinate system, transformation described in line source can be used: At time t, x-vt =X, location in moving or laser coordinate system

$x - vt' = x - vt + v\tau = X + v\tau$ and $d\tau = -dt'$

$$T-T_0 = \frac{4P}{\rho C\pi\sqrt{4a\pi}}\int_{0}^{t}\frac{d\tau(\tau)^{-0.5}}{\sigma^2+8a(\tau)}\exp[-\frac{2(X+v\tau)^2+Y^2)}{\sigma^2+8a(\tau)}-\frac{Z^2}{4a(\tau)}]$$

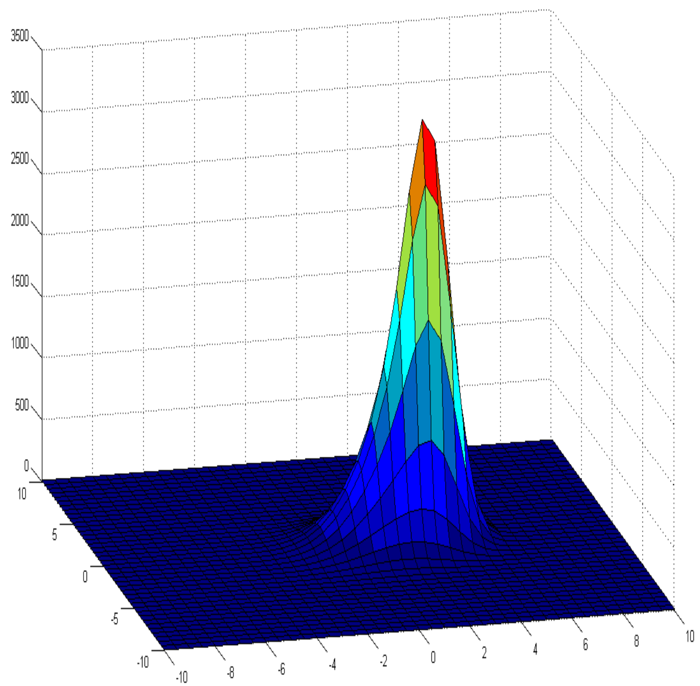
Modeling Gaussian heat source:

Material and process parameters: for EN18 steel

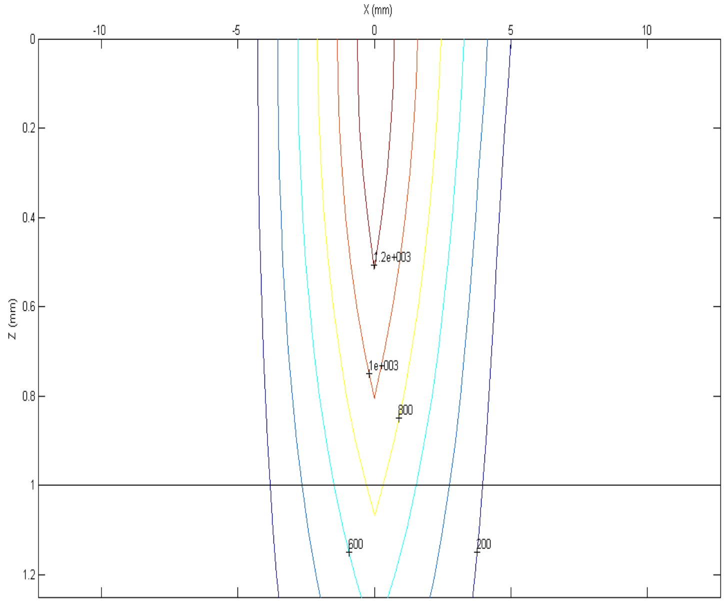
Laser power = 1300W
Scanning velocity = 100/6 mm/sec
Interaction time = 0.18sec.
Beam Radius = 1.5mm

Diffusivity = 5.1mm^2/sec
Density = 0.000008 kg/mm^3
Sp. Heat capacity = 674 J/kg k

Temperature distribution X-Y plane

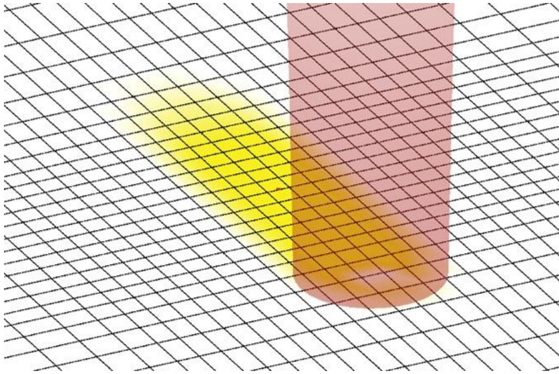


Temperature along X-Z plane.



Uniform intensity:

- Uniform circular moving heat source:



In the Uniform heat source, Q is defined by the magnitude q and the distribution parameter σ. The heat distribution, Q, is given by, $I(x,y) = \frac{P}{A}$ Where A = $\pi * \sigma^2$

for circular heat source integrating with space variables,

$$dT(X,Y,Z,t) = \frac{2Pdt'}{8\rho C\pi\sigma^2(\pi a(t-t'))^{\frac{3}{2}}}\exp[-\frac{Z^2}{4a(t-t')}]\times$$

$$\int_{-\sigma}^{\sigma}\exp[-\frac{(X-x')^2}{4a(t-t')}]dx'\int_{-\sqrt{\sigma^2-x'^2}}^{\sqrt{\sigma^2-x'^2}}\exp[-\frac{(Y-y')^2}{4a(t-t')}]dy'$$

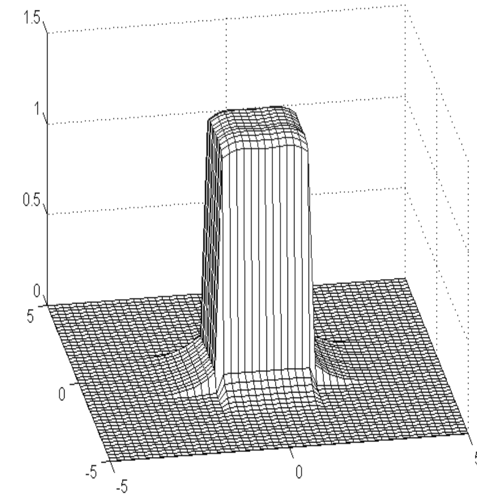
Now final temperature equation is obtained by integrating with time from 0 to t,

•Uniform rectangular moving heat source:

Rectangular heat source of dimension –l < y' < l and –b < x'< b i.e. for moving with constant velocity v from time t' = 0 to t' = t.

Heat intensity I is given by, $I(x,y) = \frac{P}{A}$ where A = 4*b*l

Integrating with the space variables,



$$dT(X,Y,Z,t) = \frac{2Pdt'}{4bl\rho C(4\pi a(t-t'))^{\frac{3}{2}}} \exp[-\frac{-Z^2}{4a(t-t')}]$$

$$\int_{-l}^{l} \exp[-\frac{(X-x')^2}{4a(t-t')}]dx' \int_{-b}^{b} \exp[-\frac{(Y-y')2}{4a(t-t')}]dy'$$

Results can be obtained by numerical integration with respect to time.

Comparison of Gaussian and uniform heat source: for EN 18 steel

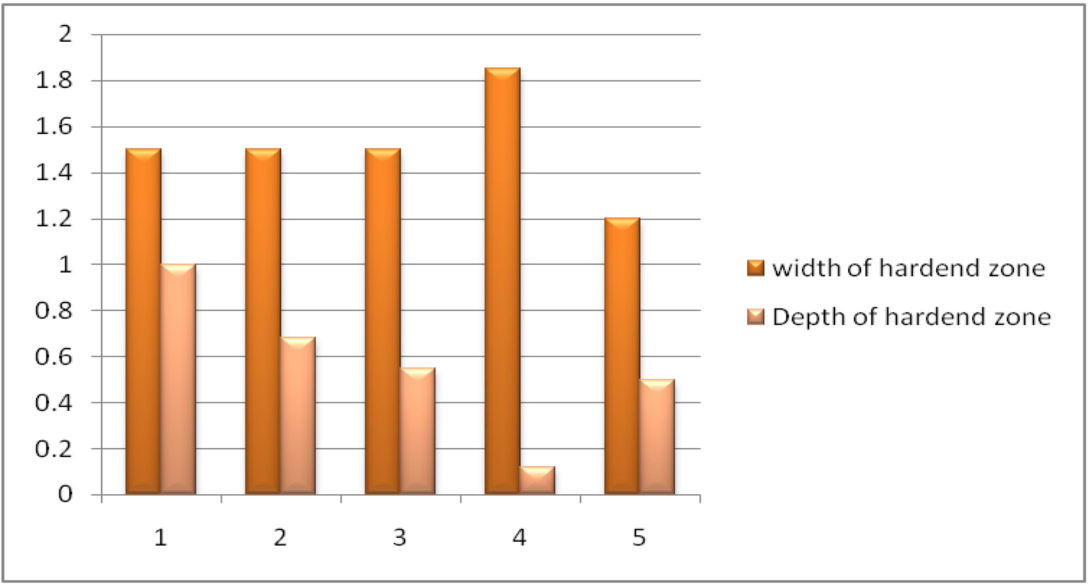
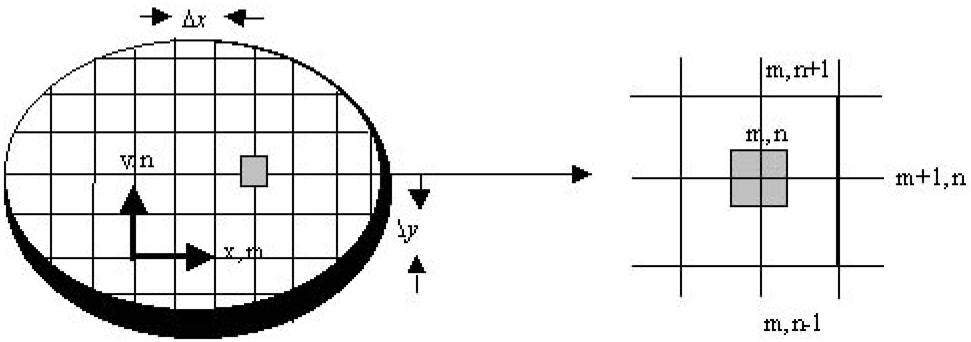


Fig. Comparison of width/depth of hardened zone[13]

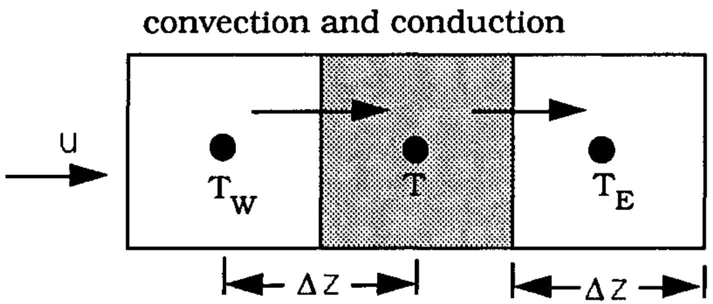
Results:

No	Distribution	Beam Shape	Max. Surface Temperature(^{0}C)	Depth (mm)	Width (mm)
1	Gaussian	Circular	1272	1	1.5
2	Uniform	Circular	1072.3	0.68	1.5
3		Rectangular Long	996	0.55	1.5
4		Rectangular short	1015.2	0.12	1.65
5		Square	1023	0.5	1.2

Finite difference formulation:



- Nodal points
- Nodal network
- Regular or irregular
- Types - coarser
  - fine



$$\alpha \frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial t} - u \frac{\partial T}{\partial z}$$

$$\alpha \left[\frac{\frac{T_w - T}{\Delta z} - \frac{T - T_E}{\Delta z}}{\Delta z} \right] \approx \frac{T' - T}{\Delta t} - u \left[\frac{T_W - T_E}{2\Delta z} \right]$$

$$\therefore\ T' = \left\{ u \left[\frac{T_W + T_E}{2\Delta z} \right] + \alpha \left[\frac{T_W + T_E - 2T}{\Delta z^2} \right] \right\} \Delta t + T$$

……Temperature at time interval Δt

Finite Element Models

Thermal Modeling

– Heat generated in workpiece due to cutting is small compared to the heat generated by the laser

– A scaled model (5mm x 2mm x 2mm) is used

– The Gaussian distribution of laser power intensity $P_{x,y}$ is given by:

$$P_{x,y} = \frac{2P_{tot}}{\pi r_b^2} exp\left(-\frac{2r^2}{r_b^2}\right)$$

– The average absorptivity of incident irradiation is determined experimentally

– Temperature dependent thermophysical properties are used

Mathematical Formulation

The 3-D transient heat conduction equation is given by,

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{Q}=\rho c_p\frac{\partial T}{\partial t}+\rho c_p V\frac{\partial T}{\partial x}$$

Initial condition,

$T(x, y, z, 0) = T_0$

Natural boundary condition on front face,

$$k\frac{\partial T}{\partial n}-q+h(T-T_0)+\sigma\varepsilon(T^4-T_0^4)=0$$

$$k\frac{\partial T}{\partial n}-q+h_e(T-T_0)=0$$

$$h_e=2.4\times10^{-3}\varepsilon T^{1.61}$$

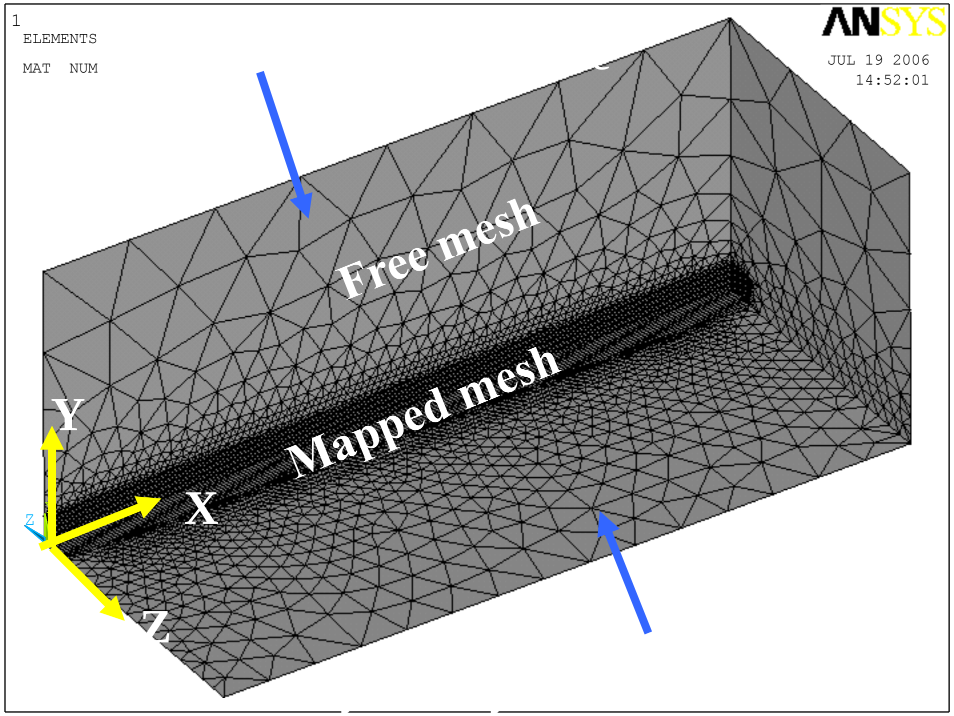
Mathematical Formulation

- Average measured temperatures are used for boundary conditions on remaining external surfaces

- Half symmetry used at bottom face

$$q_{bottom} = 0$$

Case Study- Thermal Model



- Mapped dense mesh (25 μm x 12.5 μm x 20μm)
- An 8 noded 3-D thermal element (Solid70) is used
- Gaussian distribution of heat flux applied to a 5x5 element matrix which sweeps the mesh on the front face

Temperature Simulation