Applied Mathematics: Qualifying Examination Autumn 2022

Department of Mechanical Engineering
3 pages/7 questions/100 Marks
Minumum passing marks: 40

1

Let us say there is a small opening of diameter $2r$ at the bottom of a hemispherical bowl of radius R, as shown in Fig. 1, from which water is leaking. After experiments, the relationship between the outlet velocity v, discharge coefficient C, and water level h is found to be $v = C\sqrt{2gh}$. The volume of the water in the bowl as a function of h can be taken as $V = \frac{\pi}{3}h^2(3R - h)$. You can assume at $t = 0$, the bowl is fully filled with water. Determine how long it will take for the bowl to empty. You may assume density of water to be constant. Also, you may want to use the conservation of mass. **[15 Marks]**

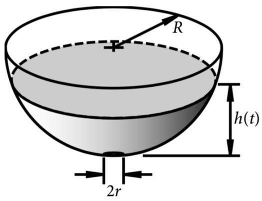

Figure 1: Water leaking problem.

2

Heat equation on a 2D infinitely long but finite width strip can be written as

$$u_t = u_{xx} + u_{yy}, \quad -\infty < x < \infty, \quad 0 < y < 1, \quad t > 0 \tag{1}$$

Initial temperature distribution is given by $u(x, y, 0)$ and is separable, i.e., $u(x, y, 0) = f(x)g(y)$. Take insulated boundaries at $y = 0$ and $y = 1$ for all $t > 0$.

1. Separate as $u(x, y, t) = v(x, t)Y(y)$. Obtain equation for $Y(y)$, its eigenfunctions, and eigenvalues. **[7 Marks]**

2. Write the differential equation for $v(x, t)$. **[2 Marks]**

3. Using suitable variable transformation by $e^{\lambda_n t}$ (here λ_n are the eigenvalues from part 1), show that the differential equation of $v(x, t)$ can be reduced to an equivalent time-dependent one-dimensional heat equation. **[5 Marks]**

4. Identify the suitable boundary conditions for part 3. **[1 Marks]**

3

The flow of current i in a LC circuit when subject to an applied voltage $v(t) = V_0 sin(\omega t)$ is given by

$$L\frac{d^2i}{dt^2} + \frac{1}{C}i = \omega V_0 cos(\omega t) \tag{2}$$

Here L denotes inductance and C denotes capacitance.

1. Obtain the general solution of current i. [**8 Marks**]
2. Can the general solution of i be approximated as an equivalent cosine wave of amplitude A_0 and frequency ω? [**2 Marks**]

4

During one of his heat transfer tutorials, Prof. Mitra assumes heat flux as

$$\mathbf{F} = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k} \tag{3}$$

1. Rahul remembers from his Maths class that heat flux has to be a conservative vector field. Prove that **F** can represent a heat flux vector field. [**5 Marks**]
2. Prof. Mitra tells the class how heat flux is related to the gradient of the temperature field as $\mathbf{F} = -\kappa \nabla T$, where κ denotes the thermal conductivity. Given that the temperature at the origin is 0, find the temperature at (2, -3, 1). [**5 Marks**]
3. Find the magnitude of heat flux along the direction **d** = x **i** + y **j** - z **k** at (0,1,-5). [**5 Marks**]

5

Consider a vector **P** = 4 **i** + **j**.

1. By what angle would you need to rotate **P** to make it coincident with its reflection about the line y=x? [**5 Marks**]
2. If $\mathbf{P}_{proj}$ represents the projection of **P** on the line y=x, find the vector representing the projection of $\mathbf{P}_{proj}$ back on **P**. What is scaling factor of this vector with respect to **P**? [**10 Marks**]

6

During an airshow the pilot decides to perform vertical loops in his plane.

1. If the ground staff are able to track the displacement of the flight in metres as

   $$\mathbf{L}(t) = (cos(t) + tsin(t))\mathbf{i} - (sin(t) - tcos(t))\mathbf{j} - (t^2)\mathbf{k} \tag{4}$$

   help them find the radius of curvature of the flight at $t = 2\,\text{sec}$. [**10 Marks**]
2. A few seconds into the flight, the pilot feels something is wrong and decides to eject seat at $t = 9.42\,\text{sec}$. Assuming the ejection is along the normal of the displacement trajectory, quantify the direction of the pilot at the time of ejection. [**5 Marks**]

7

The general solution for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given to be $\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{c}$, where $\mathbf{c}$ is $n \times 1$ column matrix containing n arbitrary constants. Using the power series expression for the exponential function, it is given that we can write

$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n(\mathbf{A})^n}{n!}.$$

1. If the matrix $\mathbf{A}$ is diagonalizable, we have the relation $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda}$, where $\mathbf{S}$ is the "modal matrix" of $\mathbf{A}$ whose columns are the linearly independent eigenvectors of $\mathbf{A}$, and $\mathbf{\Lambda}$ is the diagonal matrix with the eigenvalues of $\mathbf{A}$ as its entries. Show that $\mathbf{A}^n = \mathbf{S}\mathbf{\Lambda}^n\mathbf{S}^{-1}$. **[3 Marks]**

2. Hence show that **[3 Marks]**
$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{\Lambda}t}\mathbf{S}^{-1}.$$

3. Deduce the entries in the matrix $e^{\mathbf{\Lambda}t}$ using the expression given above for $e^{\mathbf{A}t}$. **[3 Marks]**

4. Using this procedure, solve the following system: **[6 Marks]**
$$\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} \mathbf{x}$$