Mathematics Comprehensive Examination - Spring 2022

Time: 3 hours **Total Marks: 60**

Note:

- Closed book, closed notes examination
- Respect the spirit of examination and refrain from any unfair means
- Justify each step in your solution as far as possible

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1. (a) (10 marks) Using the linear algebra techniques you have learnt, compute the volume of the parallelepiped P in $\mathbb{R}^3$ whose vertices are given by $(0,0,0),\ (3,0,0),\ (4,4,-1),\ (1,4,-1),\ (1,2,5),$ $(4,2,5),\ (5,6,4),\ (2,6,4)$. (Try to draw the geometry of the parallelepiped)

   (b) (10 marks) Generalize the method you devised in (a), to provide a procedure to compute the volume of a parallelepiped in $\mathbb{R}^n$. Specifically, consider P, a parallelepiped in $\mathbb{R}^n$, and let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \mathbf{x}_{2^n}$ be the vertices of P (P is not necessarily oriented with the principal axes in $\mathbb{R}^n$). How would you compute the volume of P using linear algebra techniques you have learnt. You just need to provide an algorithmic procedure, not a closed form solution. (Hint: In Part (a), consider the volume of one of the faces (parallelogram) of the parallelepiped in $\mathbb{R}^3$, how are the volumes of the parallelogram and that of parallelepiped related?)

2. For each continuous function $f(x,y)$ on the x,y-plane, and each path C from $(0,1)$ to $(\pi,1)$, consider the contour integral

$$\int_C y\sin^2(x)\ dx + f(x,y)\ dy$$

   (a) (7 marks) Find a choice of the function $f(x,y)$ such that the value of the above integral is independent of the choice of the path C from $(0,1)$ to $(\pi,1)$.

   (b) (3 marks) For your choice of f, evaluate the above integral for any choice of path C as above.

3. Laplace Transform of a function $f(t)$ is given as,

$$\overline{f}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

   (a) (3 marks) Use integration by parts and definition to calculate the Laplace Transform of $\int_0^t f(t')\, dt'$.

(b) (2 marks) Use the obtained result to solve the following equation,

$$\frac{du}{dt} = \int_0^t u(t')dt',$$

if $u(0) = 2$.

4. (5 marks) The modified Bessel function $I_0(x)$ is the solution to the following differential equation,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2y = 0.$$

The leading term in the asymptotic expansion of $I_0(x)$ about $x = 0$ is,

$$I_0(x) \backsim \frac{e^x}{\sqrt{2\pi x}}.$$

Assume a series form for I_0 as,

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}}\Big\{1 + b_1x^{-1} + b_2x^{-2} + \dots\Big\}.$$

Determine the coefficients b_1 and b_2.

5. The one-dimensional diffusion equation is given as,

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}, \tag{1}$$

where D is a real constant, $c(x, t)$ is a field variable, x is the position co-ordinate and t is the time.

(a) (2 marks) Show that

$$J(x, t) = D\frac{\partial c}{\partial x},$$

also satisfies the diffusion equation.

(b) (1 mark) The solution of the Eq. 1 applicable on a domain $0 \le x \le l$ for initial concentration $c(x, 0) = 0$ and boundary conditions $c(0, t) = 0$ and $c(l, t) = c_2$ is,

$$c(x,t) = c_2\frac{x}{l} + \frac{2}{\pi}\sum_{n=1}^{\infty} c_2\frac{(-1)^n}{n}\sin\Big(\frac{n\pi x}{l}\Big)\exp\Big(-\frac{Dn^2\pi^2 t}{l^2}\Big).$$

Check if the solution indeed satisfies the boundary conditions.

(c) (4 marks) The half-range periodic Fourier cosine series for an even function $f(x)$ is given as,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right). \; 0 \le x \le l$$

The coefficients a_0 and a_n can be deduced by exploiting the orthogonality of cosine and sine functions. Express

$$f(x) = \frac{x^2}{2l^2} - \frac{1}{6}, \quad 0 \le x \le l$$

as a half-range periodic Fourier cosine series.

(d) (8 marks) Use the answers to the sub-questions (b) and (c) and write down (derivation of the solution not required) the solution of diffusion equation applicable on a domain $-l \leq x \leq l$ with the following conditions,

$$c(x,0) = 0,\ D\frac{\partial c}{\partial x}\bigg|_{x=-l} = D\frac{\partial c}{\partial x}\bigg|_{x=l} = F_0.$$

Justify the written solution. Note the symmetry in the problem.

6. (5 marks) Given a differential equation,

$$\frac{d^2y}{dx^2} = -f(x),$$

applicable on domain $[0,1]$. Derive the solution $y(x)$ in terms of $f(x)$ using either Variation of Parameters or integration by parts subjected to the boundary conditions,

$$y|_{x=0} = 0, \quad \left(y + \frac{dy}{dx}\right)\bigg|_{x=1} = 0.$$

—————— **End of Paper** ——————