Department of Mechanical Engineering, IIT Bombay
Ph. D. Qualifying Examination: Applied Mathematics
January 2023
Total Questions: 9, Total points: 100, Time: 3 Hours
Minimum passing score: 40 points
Closed Book, Closed Notes Examination, TWO-A4 sheets (both sides) permitted
Fourier Transform Table on Page 3

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1. Find the general solution to the following system (homogeneous and non-homogeneous parts). Note that $x_1 \ldots x_4$ are scalar components of the column vector **x**. What is the vector in the null space of A? **(10 marks)**

$$-x_1 + 2x_2 - x_3 + x_4 = 0$$
$$2x_1 - 3x_2 - x_4 = -1$$
$$3x_1 - 5x_2 + x_3 + 2x_4 = 0$$

2. Given a matrix **A** = $\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$. Find the *orthonormal* set $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, where $\mathbf{q}_1$ and $\mathbf{q}_2$ span the column space of **A**. **(8 marks)**

3. Consider the matrix **A** = $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$. The powers $\mathbf{A}^k$ of this matrix approach a certain limit as $k \rightarrow \infty$. Given $\mathbf{A}^2 = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$, and $\mathbf{A}^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$. Using the eigenvalues and eigenvectors of **A**, show why $\mathbf{A}^2 = \frac{1}{2}(\mathbf{A} + \mathbf{A}^\infty)$? **(7 marks)**

4. A vibrating string is kept inside a periodic box simulator with dimensions [*-L, L*]. The string is given some prescribed displacement and velocity at the two ends, i.e., *-L* and *L*. The overall displacement of the string, y(t), is governed by the equation that takes the form:

$$\frac{d^2y}{dt^2} + \alpha y = 0$$

where α is a constant. Assuming periodic boundary conditions find the eigen-values and eigen functions for this configuration. **(10 marks)**

5. Convert the following equation into Sturm Liouville form (if not already in that form).

$$y'' - 2y' + \lambda y = 0$$ **(5 marks)**

6. A vector valued function in a single variable *t* is used to define the position vector, $\vec{r}$, which is used to form a curve given by:

$$\vec{r}(t) = \langle t, \ 3\sin t\,, \ 3\cos t \rangle$$

(a) Find the unit normal to this curve. **(3 marks)**
(b) Find the arc length. **(2 marks)**
(c) If a particle characterized by the function $F(x, y, z) = 8x^2yz\,\hat{\boldsymbol{\imath}} + 5z\hat{\boldsymbol{\jmath}} - 4xy\hat{\boldsymbol{k}}$ skates on this curve, calculate the line integral of *F*. **(3 marks)**
(d) Determine the acceleration of this particle **(2 marks)**

7. Solve the following coupled system of first order equations for $u(x,t), v(x,t)$ over $-\infty < x < \infty$

$$u_t + 5u_x - 4v_x = 0$$
$$v_t - 4u_x + 5v_x = 0$$

with given initial conditions $u(x,0) = f(x), v(x,0) = g(x)$.

Do NOT use integral (Fourier/Laplace, etc.) transforms to solve this problem.
Hint: diagonalize the matrix $\begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$ and then solve the uncoupled equations. Decouple the equations by using the transformation $\begin{pmatrix} w_1(x,t) \\ w_2(x,t) \end{pmatrix} = P \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$ where P is an appropriately chosen 2×2 matrix. Assume that all functions are as smooth/continuously differentiable and bounded as necessary to carry out all mathematical operations and for solutions and/or transforms to exist. **(15 marks)**

8. The Fourier transform $F(\omega)$ of $f(x)$ and its inverse are defined as follows:

$$F(\omega) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx$$

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, dx$$

Use the above definition of the Fourier transform and the accompanying transform table to solve the heat equation for $u(x,t)$

$$u_t = u_{xx}$$

over $-\infty < x < \infty$. The boundary conditions are $u(x,t), u_t(x,t) \to 0$ as $x \to \pm\infty$ and the initial condition is $u(x,0) = (1 - 2x^2)e^{-4x^2}$. Assume that all functions are as smooth/continuously differentiable and bounded as necessary to carry out all mathematical operations and for solutions and/or transforms to exist. Note: Fourier transform the x and the not the t variable. **(10 marks)**

9. In classical mechanics, a harmonic oscillator is a system that, when displaced from its equilibrium position, experiences a restoring force $F = m\frac{d^2y}{dx^2}$ proportional to the displacement y:

$$m\frac{d^2y}{dx^2} = -ky, \qquad (1)$$

where k is a positive constant. If F is the only force acting on the system, the system is called a *simple harmonic oscillator*, and it undergoes simple harmonic motion: sinusoidal oscillations about the equilibrium point, with a constant amplitude and a constant frequency (which does not depend on the amplitude). So, consider such a simple harmonic oscillator, consisting of a mass $m = 2$ kg, and experiencing a single force F, which pulls the mass in the direction of the point $y = 0$ and depends only on the position y of the mass and a constant $k = 98$ N/m.

Using ONLY the Frobenius series given by

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+s} \qquad (2)$$

solve the linear harmonic oscillator equation (1). DO NOT USE any other series form other than equation (2) **(25 marks)**

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Table of Fourier Transforms

	$f(x) = \mathcal{F}^{-1}[F](x)$	$F(\omega) = \mathcal{F}[f](\omega)$
1	$f'(x)$	$-i\omega F(\omega)$
2	$f''(x)$	$-\omega^2 F(\omega)$
3	$f(ax+b) \quad (a>0)$	$\frac{1}{a}\, e^{-i(b/a)\omega} F(\omega/a)$
4	$(f * g)(x)$	$F(\omega)G(\omega)$
5	$\delta(x)$	$\frac{1}{\sqrt{2\pi}}$
6	$e^{iax} f(x)$	$F(\omega + a)$
7	$e^{-a^2x^2}$	$\frac{1}{\sqrt{2}\,a}\, e^{-\omega^2/(4a^2)}$
8	$xe^{-a^2x^2} \quad (a>0)$	$\frac{i}{2\sqrt{2}\,a^3}\,\omega e^{-\omega^2/(4a^2)}$
9	$x^2e^{-a^2x^2} \quad (a>0)$	$\frac{1}{4\sqrt{2}\,a^5}\,(2a^2-\omega^2)e^{-\omega^2/(4a^2)}$
10	$\frac{1}{x^2+a^2} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}\,\frac{1}{a}\, e^{-a\lvert\omega\rvert}$
11	$\frac{x}{x^2+a^2} \quad (a>0)$	$-i\sqrt{\frac{\pi}{2}}\,\frac{1}{2a}\,\omega e^{-a\lvert\omega\rvert}$
12	$H(a-\lvert x\rvert) = \begin{cases} 1, & \lvert x\rvert \le a \\ 0, & \lvert x\rvert > a \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin(a\omega)}{\omega}$
13	$xH(a-\lvert x\rvert) = \begin{cases} x, & \lvert x\rvert \le a \\ 0, & \lvert x\rvert > a \end{cases}$	$i\sqrt{\frac{2}{\pi}}\,\frac{1}{\omega^2}\left[\sin(a\omega) - a\omega\cos(a\omega)\right]$
14	$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+\omega^2}$
15	$e^{-(x+b)^2/(4a)} + e^{-(x-b)^2/(4a)}$	$2\sqrt{2a}\,e^{-a\omega^2}\cos(b\omega)$
16	$\operatorname{erf}(ax)$	$i\sqrt{\frac{2}{\pi}}\,\frac{1}{\omega}\, e^{-\omega^2/(4a^2)}$