Department of Mechanical Engineering, IIT Bombay

Solid Mechanics Qualifying Examination (Spring 2020)

Instructions:

1. Total points: 50. Passing: 20.
2. Each question carries 10 points. There are total 6 questions.
3. Best 5 out of 6 will be considered for final grading. Please attempt at least five questions.
4. There are three parts. Solve the questions of each part in separate answer-books.
5. The examination is closed-books and closed-notes.
6. State your assumptions clearly. Clearly write the formulae you are using during the solution.
7. Partial points will be awarded for correct intermediate steps. Full points will be awarded only if all the answers are numerically accurate.
8. Unless otherwise stated, please ignore the effect of gravity.

Part A

(Solve in a separate answer-book.)

1. **[10 points]** A rectangular parallelepiped body occupies the region $-a \leq x_1 \leq a$, $-a \leq x_2 \leq a$ and $-b \leq x_3 \leq b$ in the current configuration. The stress tensor in the body is given by,

$$\boldsymbol{\sigma} = \frac{c}{a^2}\begin{bmatrix} -(x_1^2 - x_2^2) & 2x_1x_2 & 0 \\ 2x_1x_2 & x_1^2 - x_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where a, b and c are positive constants and $b > a$.

(a) **[3 points]** Show that $\boldsymbol{\sigma}$ satisfies the balance of linear momentum in the static case with no body force.

(b) **[3 points]** Determine the tractions that must be applied to each of the six faces of the body in order for the body to be in equilibrium.

(c) **[4 points]** The principal values (eigenvalues) of the stress tensor (principal stresses) are denoted σ_i, $(i = 1, 2, 3)$, such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. These give the maximum and minimum normal stresses at a point. It can be shown that the maximum shear stress is given by $\tau_{\text{max}} = (\sigma_1 - \sigma_3)/2$. Calculate the principal stresses of $\boldsymbol{\sigma}$ as a function of position. Then find the maximum value of τ_{max} in the full domain of the body.

2. **[10 points]** Consider the axisymmetric plane strain problem of a solid circular bar of radius a with a constant internal heat generation specified by h_0. The steady state conduction equation for temperature field $T(r)$ thus becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + h_0 = 0.$$

Using boundary condition $T(a) = T_0$, and all fields must remain bounded everywhere including at $r = 0$,

(a) **[4 points]** determine the temperature distribution, i.e., calculate $T(r)$ and

(b) **[6 points]** calculate the resulting thermal stresses σ_r, σ_θ, σ_z, and $\tau_{r\theta}$, $\tau_{z\theta}$, τ_{rz}, for the case with zero boundary stresses at $r = a$. For the axisymmetric case for circular bar, all field quantities depend only on the radial coordinate, i.e., $\sigma_r = \sigma_r(r)$, $\sigma_\theta = \sigma_\theta(r)$, $\tau_{r\theta} = \tau_{r\theta}(r)$, $T = T(r)$. So the Airy stress function $\phi = \phi(r)$ is defined by

$$\sigma_r = \frac{1}{r}\frac{d\phi}{dr}, \quad \sigma_\theta = \frac{d^2\phi}{dr^2}, \quad \tau_{r\theta} = 0.$$

Also, the compatibility condition for the plane strain problem is

$$\nabla^4\phi + \frac{E\alpha}{1-\nu}\nabla^2 T = 0,$$

which can be used for calculating the thermal stresses, where E is the Elastic modulus, α is the coefficient of thermal expansion and ν is Poisson's ratio.
Hint: When $T = T(r)$, for the axisymmetric plane strain problem, prove that

$$\nabla^2 T = \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr},$$
$$\nabla^4\phi = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\right]\right\},$$

where, in polar coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$
$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right).$$

Such solutions are useful to determine the thermal stresses in rods made of radioactive materials.

Part B

(Solve in a separate answer-book.)

3. **[10 points]** An Euler-Bernoulli beam of constant flexural rigidity EI is simply supported at its ends x = 0 and x = L and has a uniformly distributed load q (per unit length) acting on it as shown in Fig. 1. Use the minimum potential energy theorem to determine the transverse

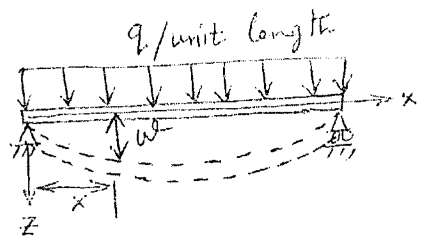

Figure 1: An Euler-Bernoulli beam

deflection w of the beam at x = L/2 for the following two cases:

(a) **[4 points]** Assume $w(\mathrm{x}) = \mathrm{a_1 x(L - x)}$, where $\mathrm{a_1}$ is an unknown coefficient.

(b) **[4 points]** Assume $w(\mathrm{x}) = \mathrm{a_1} \sin(\pi \mathrm{x/L})$, where $\mathrm{a_1}$ is an unknown coefficient.

(c) **[2 points]** Compare the solutions with the exact solution:

$$w(\mathrm{x = L/2})_{\text{exact}} = \frac{5\mathrm{qL}^4}{384\mathrm{EI}},$$

and obtain percentage errors in both cases.

4. **[10 points]** Consider a solid body subject to small deformations. Let $\mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$ be the unit orthonormal Cartesian base vectors (Canonical basis) along x, y and z coordinate axes, respectively. Consider three points P, Q and R in the body. The position vectors of these points before deformation are $\mathbf{r_P} = 2\mathbf{e_1} + 2\mathbf{e_2} + 3\mathbf{e_3}$, $\mathbf{r_Q} = 4\mathbf{e_1} + 4\mathbf{e_2} + 5\mathbf{e_3}$ and $\mathbf{r_R} = 5\mathbf{e_1} + 5\mathbf{e_2} + 3\mathbf{e_3}$. The following displacement field is imposed on the body:

$$\boldsymbol{u} = k(x^2 + y)\mathbf{e_1} + k(y + z)\mathbf{e_2} + k(x^2 - 2z^2)\mathbf{e_3}, \quad \text{where, } k = 10^{-3}.$$

(a) **[3 points]** Using the strain-displacement relations, evaluate all the small strain tensor components at point P.

(b) **[2 points]** Consider the linear material elements PQ and PR. Find the extensional (longitudinal) strains ϵ_{PQ} and ϵ_{PR} at point P in the directions of PQ and PR, respectively.

(c) **[3 points]** After deformation, the points P, Q and R occupy new positions P', Q' and R', respectively. Find the orientations of the deformed line segments $P'Q'$ and $P'R'$ using ϵ_{PQ} and ϵ_{PR}.

(d) **[2 points]** Find the angle θ between the original undeformed line segments PQ and PR. Also, find the angle θ' between the deformed line segments $P'Q'$ and $P'R'$ using ϵ_{PQ} and ϵ_{PR}.

Part C

(Solve in a separate answer-book.)

5. **[10 points]** A long cylindrical solid shaft with cross-sectional diameter D has Young's modulus E, shear modulus G and uniaxial tensile yield stress Y. The shaft carries an axial torque T, axial force P and bending moment M.

   (a) **[5 points]** Work out and write down the maximum tensile stress and the maximum shear stress in the shaft in terms of the given quantities.

   (b) **[5 points]** Use the Tresca (maximum shear stress theory) yield criterion to determine the combination of P, T, M at which the failure of the shaft by plastic deformation is just initiated.

6. **[10 points]** Consider a long elastic beam/column of length L and flexural rigidity EI. The beam is held fixed into the wall at one end and is free at the other end. An axial compressive force P and a bending moment M_0 are applied at the free end as shown in Fig. 2.

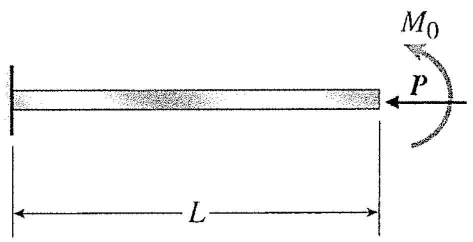

Figure 2: A beam of length L subject to an axial compressive force P and a bending moment M_0 at the free end.

   (a) **[2 points]** Set up the governing second-order differential equation to find the buckling load P_c of the beam.

   (b) **[4 points]** Solve the differential equation to obtain the transverse deflection of the beam for a given axial load $P < P_c$ and moment M_0.

   (c) **[4 points]** Determine the maximum bending moment in the beam for a given axial load $P < P_c$ and moment M_0.