PhD Qualifying Examination in Solid Mechanics [Jan 2019]
Department of Mechanical Engineering, IIT Bombay
Total points: 60

Instructions:

- Examination is closed book, closed notes.
- **Solve every question on a separate answer sheet.**
- Clearly mention any assumptions you make.
- *For questions involving numbers:* Full points will be awarded only if all the answers are numerically accurate. Partial points may be awarded if the numerical answers are wrong but the formulae used or procedure followed is correct. For all steps, first clearly write the formulae you are using during the solution.

Q1. Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be the unit orthonormal Cartesian base vectors (canonical basis). At a particular point in a body, the traction vectors on different planes are as follows:
$\mathbf{t}(\mathbf{n}) = 4\mathbf{e}_1$ when $\mathbf{n} = \mathbf{e}_1$; $\mathbf{t}(\mathbf{n}) = -3\sqrt{3}\mathbf{e}_2 - 5\mathbf{e}_3$ when $\mathbf{n} = \mathbf{e}_3$; and $\mathbf{t}(\mathbf{n}) = (4/\sqrt{6})\,\mathbf{e}_1 + [(2-3\sqrt{3})/\sqrt{6}]\,\mathbf{e}_2 - [(6\sqrt{3}+5)/\sqrt{6}]\,\mathbf{e}_3$ when $\mathbf{n} = (1/\sqrt{6})(\mathbf{e}_1 + 2\,\mathbf{e}_2 + \mathbf{e}_3)$.

a) Find the traction vector $\mathbf{t}(\mathbf{n})$ when $\mathbf{n} = \mathbf{e}_2$? [2]

b) Find the Cauchy stress tensor $\boldsymbol{\sigma}$ at this point. [1]

c) Find the magnitude of normal stress T_n and the magnitude of shear stress T_s on a plane with normal $\mathbf{m} = (1/\sqrt{3})(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ at the given point. [1]

d) Find the principal stresses and principal directions ($\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$) of the Cauchy stress $\boldsymbol{\sigma}$ at this point. [3]

e) Let a new set of orthonormal basis vectors ($\mathbf{e}_1^*$, $\mathbf{e}_2^*$, $\mathbf{e}_3^*$) coincide with the principal directions ($\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$) of the stress tensor $\boldsymbol{\sigma}$. Find the orthogonal tensor $\mathbf{Q}$ that rotates vectors from ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) basis to ($\mathbf{e}_1^*$, $\mathbf{e}_2^*$, $\mathbf{e}_3^*$) basis. [1]

f) Using the above tensor $\mathbf{Q}$ and stress tensor $\boldsymbol{\sigma}$ obtained above, find the matrix of components of the stress tensor $\boldsymbol{\sigma}^*$ in the ($\mathbf{e}_1^*$, $\mathbf{e}_2^*$, $\mathbf{e}_3^*$) basis. Comment on the form/structure of $\boldsymbol{\sigma}^*$. [2]

Q2. Consider a solid body subjected to small deformations. Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be the unit orthonormal Cartesian base vectors (Canonical basis) along x, y and z coordinate axes, respectively. Consider two points P, Q in the body infinitesimally close to each other. The position vector of point P is $\mathbf{r}_P = 2\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$, while the direction cosine of line segment PQ is $(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$. The following displacement field is imposed on the body:

$$\mathbf{u} = k(x^2 + y^2)\mathbf{e}_1 + k(y^2+z^2)\mathbf{e}_2 + k(x^2+2z^2)\mathbf{e}_3, \text{ where } k = 10^{-3}.$$

After deformation, the points P, Q occupy new positions P', Q', respectively.

a) Using the strain-displacement relations, evaluate all the small strain tensor components at point P. [2]

b) Evaluate $\Lambda = [(P'Q')^2-(PQ)^2]/(PQ)^2$. Note that PQ and P'Q' denote the straight line distance between points P, Q and P', Q' respectively. [3]

c) If $P'Q' = PQ(1+\varepsilon_{PQ})$ then express ε_{PQ} in terms of Λ and solve for ε_{PQ} ignoring any 2nd and higher order terms in ε_{PQ}. [2]

d) Find the direction cosines or orientation of the deformed line segments P'Q'. [3]

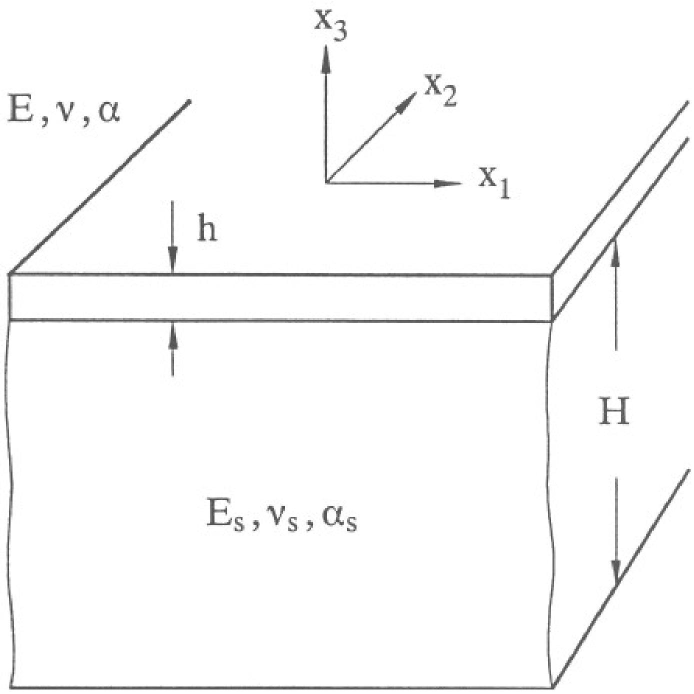

Fig. 1. A thin film on a thick substrate (h<<H)

Q3. A thin film of material (modulus E, Poisson's ratio ν, and α = Coeff. of thermal expansion) is bonded to a thick substrate (modulus E_s, Poisson's ratio ν_s, and α_s = Coeff. of thermal expansion) at temperature T_0 as shown in Fig. 1. Assume elastic behaviour, and no initial stresses post bonding at T_0.

a) The temperature is then changed to T. Calculate the resultant stress state on the thin film (assume very large interface area, estimate behaviour away from edges). [6]

b) If $\alpha_s > \alpha$, and $T > T_0$, and the bonding is very strong and uniform, what is the potential failure mode of thin film away from edges. Briefly explain. [2]

c) If $\alpha_s > \alpha$, and $T < T_0$, and the bonding is <u>not</u> very strong and uniform, what is the potential failure mode of thin film away from edges. Briefly explain. [2]

Q4. A helical spring shown in Fig. 2a has coil diameter as D, wire cross sectional area equal to a circle of diameter d, and number of coils N. Assume linear elastic isotropic material properties (Young's modulus E, Poisson's ratio ν).

Fig. 2a. A helical spring 2b. A slinky toy 2c. A freely suspended slinky toy

a) What loading states: axial, bending, pure torsion, direct shear – are present on any vertical cross-section of the coil wire, when an external load is applied at its free end? [1]

b) Which loading state will account for majority of the deflection of the spring? [1]

c) Considering only the major loading state (from (b)), and ignoring other loading states & gravity, calculate the stiffness of the spring. [4]

d) A slinky toy (Fig 2b) is essentially a low stiffness helical spring using a thin cross section wire (d) and large number (N) of coils (coil-diameter D). Assume the slinky (Fig 2c) is suspended from one end under its own weight (Young's modulus E, Poisson's ration ν, Density ρ), and achieves equilibrium (i.e. it is not oscillating). Calculate the extended length of the spring. Assume small strains. [4]

Q5. A thin square plate containing a hole of radius a and subjected to equi-biaxial stress σ_0 is under static equilibrium as shown in Fig. 3. The hole centre coincides with the plate centroid and compared to a, the plate dimensions are much bigger such that for all practical purposes the plate can be regarded as an infinite plane. Assuming plate as linear elastic isotropic, use the thick cylinder stress solution given below and compute the stress field in the plate. [6]

Stress field in a thick linear elastic, isotropic cylinder of inner radius r_1 and outer radius r_2 subjected to pressure p_1, p_2 respectively:

$$\sigma_r = \frac{\left(p_1 r_1^2 - p_2 r_2^2\right)}{\left(r_2^2 - r_1^2\right)} - \frac{r_1^2 r_2^2}{r^2} \frac{\left(p_1 - p_2\right)}{\left(r_2^2 - r_1^2\right)} \qquad (1)$$

$$\sigma_\theta = \frac{\left(p_1 r_1^2 - p_2 r_2^2\right)}{\left(r_2^2 - r_1^2\right)} + \frac{r_1^2 r_2^2}{r^2} \frac{\left(p_1 - p_2\right)}{\left(r_2^2 - r_1^2\right)} \qquad (2)$$

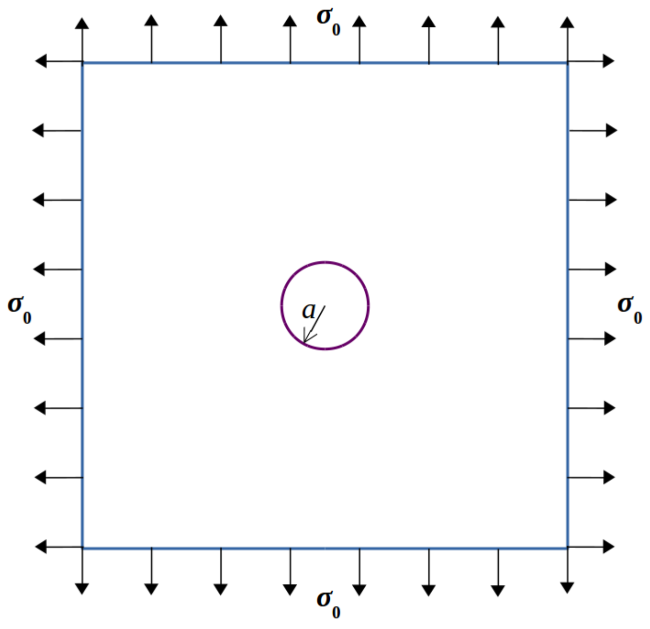

Fig. 3

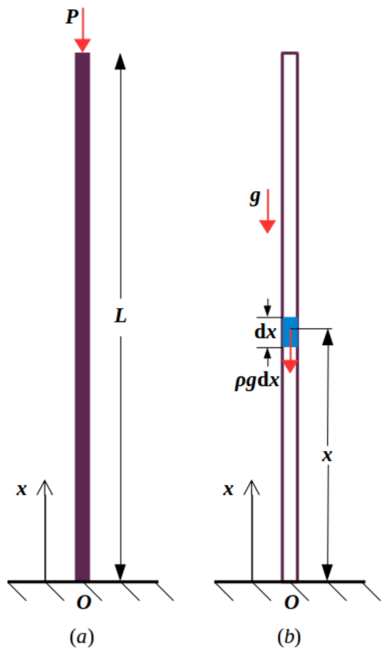

Fig. 4

Q6. Total potential energy Π of an elastic system is the sum of the stored strain energy Π_s and the potential energy of the external forces Π_{load}. *Principle of minimum potential energy* states that static equilibrium of an elastic body implies that the total potential energy Π must be minimum with regards to any kinematically admissible small variation in the displacement field. Here kinematically admissible displacement field means a displacement field which is single-valued, continuous and satisfies the displacement boundary conditions of the problem under consideration.

For a linear elastic homogeneous column of length L, bending stiffness EI clamped at one end and subjected to axial load P at the free end (see Fig. 4a), Π is given as,

$$\Pi[v(x)]=\int_0^L \frac{1}{2} EI\left(\frac{d^2 v(x)}{dx^2}\right)^2 dx-P\int_0^L \frac{1}{2}\left(\frac{dv(x)}{dx}\right)^2 dx \qquad (3)$$

where $v(x)$ is the horizontal (transverse) displacement of the column. The first term on the R.H.S of Equation (3) is the stored strain energy while the second term is the potential energy of the external force. Note that the integral in the second term is the vertical displacement (x-component) of the free end.

(a) Give reason why $v(x)=Cx^2$, with C being an undetermined constant, is a kinematically admissible displacement field for the column in Fig. 4a. [2]

(b) Use $v(x)$ from (a) and compute Π. [2]

(c) Minimize Π with respect to C, *apply* Principle of minimum potential energy and compute the critical load for buckling. [3]

The same methodology (a-c) can be applied to compute the critical load for buckling due to self-weight. Assume the column cross-section to be A, mass density ρ, acceleration due to gravity g (see Fig. 4b).

(d) Modify Equation (3) if the column is stressed owing to self-weight instead of axial load P. [2]

(c) Use approach of (a-c) to obtain the critical length for preventing buckling due to self-weight. [5]