PhD Qualifying Examination
DES 1: Solid Mechanics
Spring 2023

Instructions:

1. There are **6** questions in this paper. All questions will be graded and partial marks will be awarded wherever applicable.
2. Maximum Marks: **100**, Passing Marks: **40**.
3. The examination is closed-books and closed-notes. Use of a non-programmable scientific calculator is permitted.
4. If any part of the question is incomplete, incorrect, or inconsistent, please make the necessary and appropriate assumption(s) to solve the problem. Include the appropriate justification for any assumption you make while solving the problems.

Question 1

Consider a solid subjected to body loads and surface tractions that is in equilibrium (there are no internal couples in the solid). The traction vectors on two different planes at a particular point in the body are

$$\mathbf{t}(\mathbf{n}) = -5\mathbf{e}_1 - 3\sqrt{3}\mathbf{e}_2 \quad \text{for} \quad \mathbf{n} = \mathbf{e}_1,$$
$$\mathbf{t}(\mathbf{n}) = 4\mathbf{e}_3 \quad \text{for} \quad \mathbf{n} = \mathbf{e}_3,$$

where $\mathbf{e}_i$ $(i = 1, 2, 3)$ are the canonical orthonormal Cartesian base vectors.

(a) Determine all possible components of the Cauchy stress tensor ($\boldsymbol{\tau}$) from the tractions given above. Can you determine the complete stress tensor? **(4 marks)**

(b) The characteristic equation for principal stresses of the stress tensor is of the form

$$\lambda^3 + a\lambda + b = 0,$$

where a and b are constants. Determine the principal stresses and principal planes at this point in the solid. What is the value of the maximum shear stress? **(5 marks)**

(c) Determine the magnitude of normal and shear stresses on the octahedral plane (the plane with normal $\mathbf{n} = \{1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\}$ with respect to the principal planes). **Do not write down the formula from memory. You have to derive the formula from the given information** **(3 marks)**

(d) Consider the orthonormal coordinates given below:

$$\mathbf{e}_1^{'} = \frac{\sqrt{3}}{4}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \frac{\sqrt{3}}{2}\mathbf{e}_3$$
$$\mathbf{e}_2^{'} = \frac{3}{4}\mathbf{e}_1 + \frac{\sqrt{3}}{4}\mathbf{e}_2 - \frac{1}{2}\mathbf{e}_3$$
$$\mathbf{e}_3^{'} = -\frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2$$

(i) What is the transformation ($\mathbf{Q}$) between the coordinate systems given by the basis vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$ and $\mathbf{e}_i'$ $(i = 1, 2, 3)$?

(ii) What is the expression for the components of the stress tensor ($\boldsymbol{\tau}'$) in the coordinate basis e_i' as a function of $\mathbf{Q}$ and $\boldsymbol{\tau}$? Is it $\boldsymbol{\tau}' = \mathbf{Q}^T\boldsymbol{\tau}\mathbf{Q}$ or $\boldsymbol{\tau}' = \mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T$?
Hint: You may want to justify your choice of expression by considering the representation of a traction vector in the coordinate systems given by the basis $\mathbf{e}_i$ *and* $\mathbf{e}_i'$.

(iii) Determine the stress tensor in the orthonormal coordinates $\mathbf{e}_i'$.

(4 marks)

Question 2

Consider a tapered column of length L as shown in fig. 1(a). The Young's Modulus for this column is E, and the cross section is circular, with the radius linearly varying from R_o at the bottom to R_i at the top (assume that the length is much longer than the largest radius). The column is subjected to gravity loading, as shown in fig. 1. The goal of this problem is to find the critical length of the column when buckling occurs due to self-weight.

(a) Write down the Potential Energy for the column assuming lateral deflection due to buckling.
Note: The axial contraction of a beam of length L subjected to a transverse deflection w can be obtained as $\Delta = \int_0^L \left(\frac{dw}{dy}\right)^2 dy$, *where y is the axial coordinate along the length of the beam.* **(3 marks)**

(b) In order to estimate the critical length of the column for buckling, assume the lateral deflection to be approximated by a **quadratic polynomial**. What are the conditions that the approximate solution should satisfy? **(3 marks)**

(d) Using the **Principle of Minimum Potential Energy**, determine the (approximate) critical length of the column to avoid buckling. Assume $R_i/R_0 = 0.5$. **(5 marks)**

(e) If the column is inverted, i.e., the smaller cross-section end is fixed at the bottom (as shown in fig. 1(b)), will the critical length of the column for buckling increase or decrease? Determine the critical length for this case (use $R_i/R_0 = 0.5$). **(2 marks)**

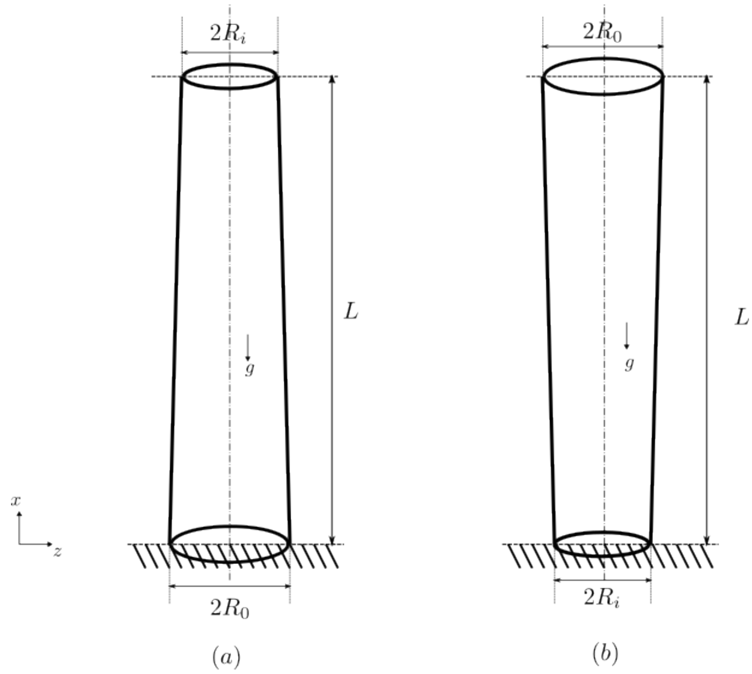

Figure 1: Figure for Question 2: A tapered column subjected to gravity loading.

(f) In order to ensure that both configurations shown in fig. 1 have the same critical buckling load for a given length (the larger critical length you have determined), a spring is attached to the weaker column in the lateral direction (along z). Will this solution work as intended? Provide a qualitative justification based on the analysis done above. **(3 marks)**

3. Figure 3 shows the square cross-section of a solid undergoing plane strain deformation with the following state of in-plane strain:

$$\varepsilon_{xx} = Ay^3, \varepsilon_{yy} = Ax^3, \varepsilon_{xy} = Bxy(x+y),$$

where A, B are positive real constants.

a) Find out the relationship between *A* and *B*. **(3 marks)**

b) Find out the in-plane components of displacement if the edges *x* = 0 and *y* = 0 are fully constrained. **(6 marks)**

c) A marker is located at (0.5, 0.5) in the undeformed configuration as shown in Figure 3. Calculate its co-ordinate after the deformation. **(3 marks)**

d) Calculate the strain recorded by strain gages are located at (0.5, 0.5) inclined at an angle of 45° and 135° with respect to positive *x*-axis (See Figure 3). **(4 marks)**

e) Calculate the change in angle between the strain gages following deformation. **(2 marks)**

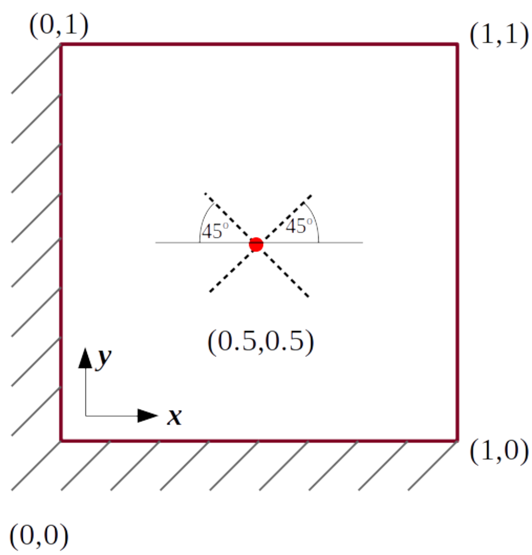

Figure 3: Problem 3

4. As shown in Figure 4, a linear elastic disc under diametrical compression can be decomposed into three parts viz. Flamant Solution (1), Flamant Solution (2) and Radial Tension Solution (3). The Airy stress function corresponding to the Flamant Solution (1) is given as,

$$\phi_1(r_1, \theta_1) = -\frac{P}{\pi} r_1 \theta_1 \sin\theta_1$$

where P is the force per unit out-of-plane thickness. r_1, θ_1 is as described in the image corresponding to Flamant Solution (2) in Figure 4. Disc has uniform thickness and diameter D.

a) Write down the boundary conditions for the disc shown in the first image of Figure 4. **(2 marks)**

b) Express the Airy stress function ϕ_1 given above in terms of cartesian x-y co-ordinate system as shown in the disc image of Figure 4. Origin O coincides with the centre of the circular disc. **(3 marks)**

c) Calculate the in-plane components of stresses σ_{xx}, σ_{yy}, σ_{xy} for Flamant Solution (1) using the answer to previous sub-question. **(6 marks)**

d) Use the answers to previous two sub-question and estimate the solution to Flamant Solution (2). **(4 marks)**

e) Does the combination of Flamant Solution (1) and Flamant Solution (2), satisfy the boundary conditions for the disc as noted in sub-question (a)? Justify your answer. If not, find out the radial traction distribution $t_r(\theta)$ to be applied to the disc boundary as depicted in the Radial Traction Solution (3) to ensure that the boundary conditions as stated in sub-question (a) are ensured. **(5 marks)**

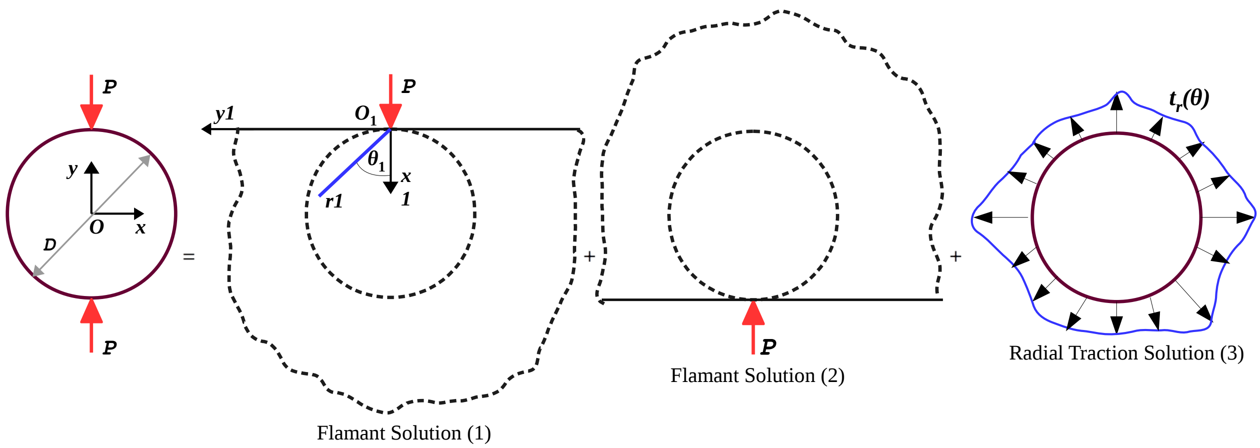

Figure 4: Problem 4

5 Consider a composite cylindrical shaft made of two different materials as shown in the figure.

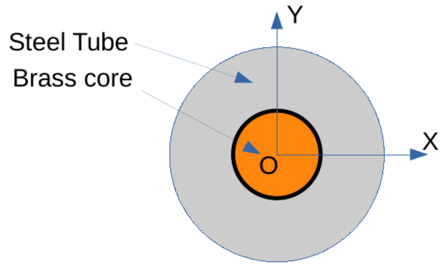

Figure 5: Problem 5

The tubular outer portion is made of steel (outer radius 50 mm, inner radius 20 mm) and its inner core (radius 20 mm) is made up of brass. The brass core is perfectly bonded to the steel tube. Assume that both steel and brass behave as linear elastic, perfectly plastic materials. The material properties are as follows:

	Shear Modulus (GPa)	Yield Strength in Shear (MPa)
Steel	70	300
Brass	40	250

Table 1: Problem 5

(a) Determine the maximum torque, T_e, that can be carried by the composite shaft before either the steel tube or the brass core yields. **(7 marks)**

(b) Now assume that the torque acting on the composite shaft, T, is increased gradually from T_e till the critical part, i.e. the part that yields first (identified in part (a)) undergoes complete plastic deformation. Find this torque T. **(8 marks)**

6 Consider the structure made up of a cantilever beam, a 'L' shaped beam and a massless rigid rod as shown in the figure.

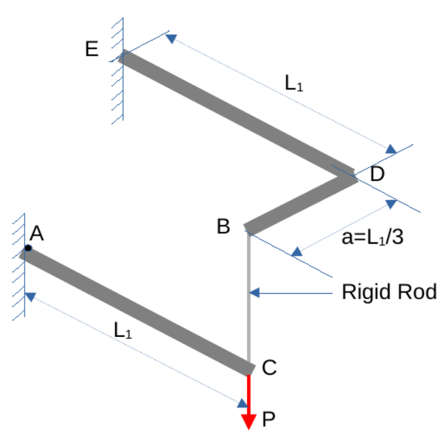

Figure 6: Problem 6

The 'L' shaped beam and the cantilever beam have a circular cross-section with radius r. The top surface of the cantilever beam is connected to a 'L' shaped beam with a rigid massless rod. A vertical force P acts at the center of the cross-section of the free end of the cantilever beam. The 'L' shaped beam and the cantilever beam are made of the same material with Young's modulus E and Shear modulus G. Assume that the deflections and rotations are small.

(a) Find the normal stress at point A which is located on the top surface of the cantilever as shown in the figure. **(6 marks)**

(b) Now consider the case when the rigid massless rod is replaced by a linear massless spring with spring constant k. The unstretched length of the spring is equal to initial separation between points B and C, i.e. before the load P is applied. Find the normal stress, σ_A at point A. **(7 marks)**

(c) Find $\lim_{k\to 0} \sigma_A$ and $\lim_{k\to\infty} \sigma_A$. **(2 marks)**

Hint: The net deflection of point B in the vertical direction has two components: rigid body motion and deformation. The rigid body motion is due to the bending and twisting of portion ED while the deformation is due to the bending of portion BD.
Note that for a cantilever beam with length L, flexural rigidity EI and loaded by a vertical point force F at the free end, the vertical deflection of the free end in the direction of the force is given by

$$\delta = \frac{FL^3}{3EI}$$

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