PhD Qualifying Examination
DES 1: Solid Mechanics
Spring 2022

Instructions:

1. There are a total of 6 questions. Each question carries 10 marks. All questions will be graded and partial marks will be awarded wherever applicable.
2. Maximum Marks: 60, Passing Marks: 24.
3. There are two parts in the paper: A, B. Please solve each part on separate sheets/answer books.
4. The examination is closed-books and closed-notes. Use of a non-programmable scientific calculator is permitted.
5. If any part of the question is incomplete, incorrect, or inconsistent, please make the necessary and appropriate assumption(s) to solve the problem. Include the appropriate justification for any assumption you make while solving the problems.

Part A

Question A1

Consider the compound bar that is axially loaded as shown in fig. 1. The cross-section of the bar is circular (solid/hollow), and the dimensions of the individual sections, along with their Young's Modulus are given in Table 1. The magnitude of the forces applied on the bar are

$$F_1 = 1500\,\text{kN}, \quad F_2 = 850\,\text{kN}, \text{ and } \; F_3 = 650\,\text{kN}.$$

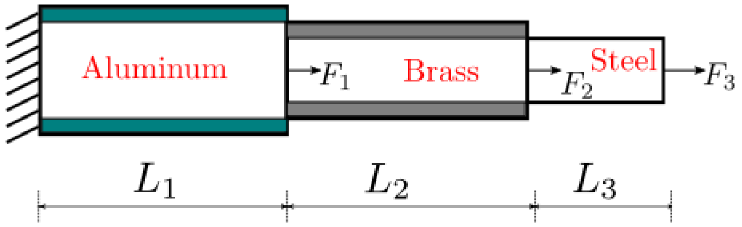

Figure 1: A compound bar loaded by axial force.

Material	Length (m)	Outer Diameter (m)	Inner Diameter (m)	Young's Modulus (GPa)
Aluminum	1	0.2	0.125	73
Brass	1.25	0.15	0.1	100
Steel	0.75	0.1	0	210

Table 1: Geometry and Material Properties for the Compound bar shown in fig. 1.

(*a*) Assuming that the only possible deformation is along the axis of the bar (you can neglect all other deformations, as well as non-idealities associated with sharp corners/transitions etc.), use the **Principal of Minimum Potential Energy** to determine the axial displacement of each individual component of the bar. (**7 marks**)

(*b*) Apart from 1-D deformation, what are the implicit assumptions involved in determining the solution? Suppose,the bar is now made of materials whose Young's modulus is $(1/1000)^{\text{th}}$ of the values given above (in other words, the magnitude is the same but the units are MPa instead of GPa). Can you use the solution determined above to determine the displacements for this bar (the forces applied are not changed)? If not, what changes do you need to make to determine the solution? Are there any other limitations to the modifications you suggest? (Qualitative answers will suffice) (**3 marks**)

Question A2

Consider a tapered cantilever beam of length L as shown in fig. 2. The cross section is circular, and the radius gradually varies from R_o at the left end, to R_i at the right end ($R_0 > R_i$; assume that the length is much longer than the largest radius). The Young's Modulus for this beam is E, and a compressive force of magnitude F is applied on the right end (at the centroid of the cross-section; the left end is fixed), as shown in fig. 2.

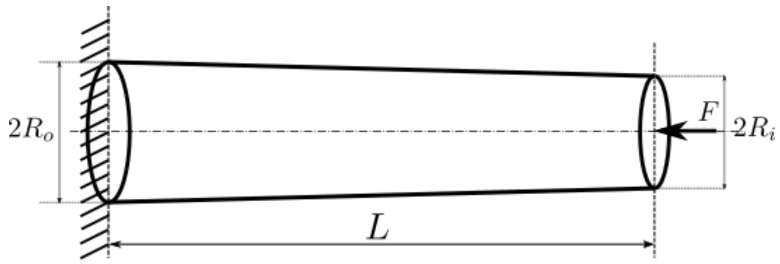

Figure 2: A tapered cantilever beam subjected to an end load.

(a) Draw the free body diagram, and determine the governing differential equation for the beam. (**2 marks**)

(b) What are the boundary conditions for the beam? Determine the characteristic equation for the critical load(s) at which buckling happens in this beam. (**4 marks**)
Hint: The solution to the differential equation $x^4y'' + y = 0$ *is* $y = x\sin(1/x)$. $(y' = dy/dx)$.

(c) If the beam is fixed at the right end, and a compressive load is applied at the left end, will the value of the critical load change (in other words, the beam is fixed at the smaller cross-section (R_i), and the load is applied on the larger cross-section (R_0))? Give a qualitative answer with the appropriate justification. (**2 marks**)
Hint: Try to obtain a "graphical" solution for the characteristic equation, in case you are not able to find an analytical solution.

(d) If the beam is **hinged at both ends**, instead of the cantilever case shown in fig. 2, would the buckling load depend on which end it is applied at? Please provide appropriate justification for your answer. (**2 marks**)

Question A3

A square plate with a circular hole at the center is subjected to a uniform, uniaxial tensile load (T) as shown in fig. 3. The length of the plate is much larger than the radius (a) of the hole. The governing differential equation for this problem can be written as

$$\nabla^4\phi = 0,$$

where ϕ is the Airy stress function. For this particular scenario, the Airy stress function can be chosen as

$$\phi = f(r) + g(r)\cos 2\theta,$$

where

$$f(r) = c_0 + c_1 r + c_2 r^2 + c_3 r^2 \log r \quad \text{and} \quad g(r) = d_0 + d_1 r^2 + d_2 r^4 + d_3 r^{-2},$$

where r, θ are the polar coordinate variables, whose origin coincides with the center of the plate, and c_i, d_i $(i = 0, 1, 2, 3)$ are constants that need to be determined.

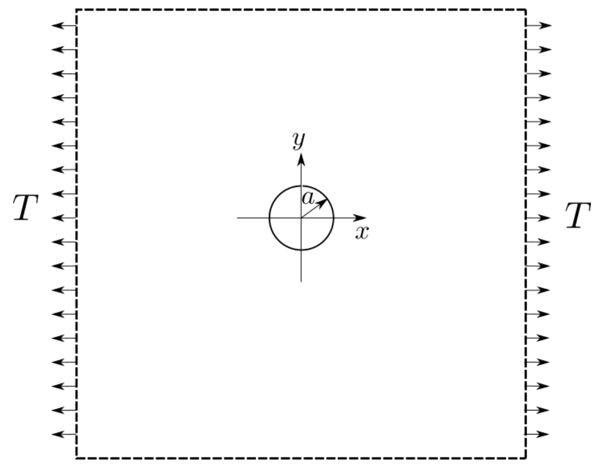

Figure 3: A large square plate with a small hole, subjected to uniaxial tension.

(a) What "implicit" assumption do you need to make to solve this problem? Consequently, what are the boundary conditions you will need to solve for the constants in the Airy Stress function? (**2 marks**)

(b) Using the boundary conditions, determine the stress field in the plate as a function of T (relevant formulae provided at the end). (**3 marks**)

(c) Where is the maximum stress on the hole? What is the stress concentration factor (with respect to T)? (**2 marks**)

(d) If the loading is changed as shown in fig. 4, would you be able to use the results determined above to solve this new loading condition? If so, please write down a proper justification for the same, and the solution framework. **(3 marks)**

Hint: Try to relate the given loading condition to an equivalent loading condition such that you can leverage the solution you determined in (a)-(c).

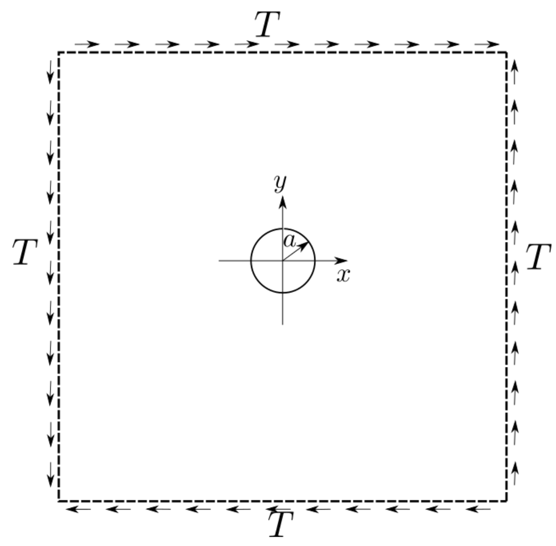

Figure 4: A large square plate with a small hole, subjected to shear loading.

Relevant Formulae:

1. The relationship between the components of the stress tensor in the Cartesian and Polar coordinates can be expressed as

$$\begin{aligned}
\sigma_r &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\
\sigma_\theta &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\
\tau_{r\theta} &= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta
\end{aligned}$$

2. The components of the stress tensor in polar coordinates can be written in terms of the Airy Stress function as

$$\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\
\sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} \\
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)
\end{aligned}$$

Part B

Please write the solutions to Part B on a separate answer sheet.

B1. Consider a thick open cylinder of length L with inner radius a and outer radius $2a$ as shown in the figure. **(10 marks)**

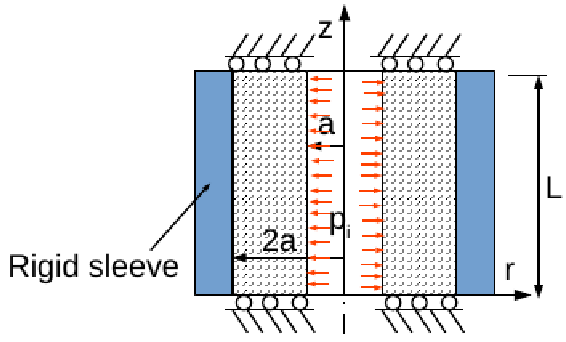

Figure 1: Problem 1

The cylinder is placed in a rigid frictionless sleeve which prevents the radial displacement of the outer radius. The cylinder is also prevented from expanding along the axial direction as shown. It is subjected to an internal pressure p_i. The cylinder is made up of isotropic homogeneous material with Young's modulus E and Poisson's ratio v. It has a yield stress σ_Y in uniaxial tension.

(a) State the boundary conditions at the inner radius, outer radius, the top face ($z = L$) and the bottom face ($z = 0$).

(b) If the radial displacement is governed by the following ordinary differential equation:

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$

find the stress components σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}. Plot the variations of the stress components as function of the radial coordinate r.

(c) Where and at what internal pressure will the yielding occur if the material obeys the Tresca's yield criteria.

(d) Now consider the same problem with $\nu = 0$. If the temperature of the cylinder is uniformly increased by ΔT, do you anticipate any change in the pressure that causes yielding. You need to justify your answer.

Note: The equilibrium equations in absence of body forces in cylindrical coordinates are given by:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} = 0$$

B2. Consider a built-in bent solid rod made of segments OA, AB and BC as shown in the figure. A vertical force P acts at the free end at point C. The rod is made of elastic isotropic material with Young's modulus E and shear modulus G. The rod has a circular cross-section with moment of inertia A and polar moment of inertia J. **(10 marks)**

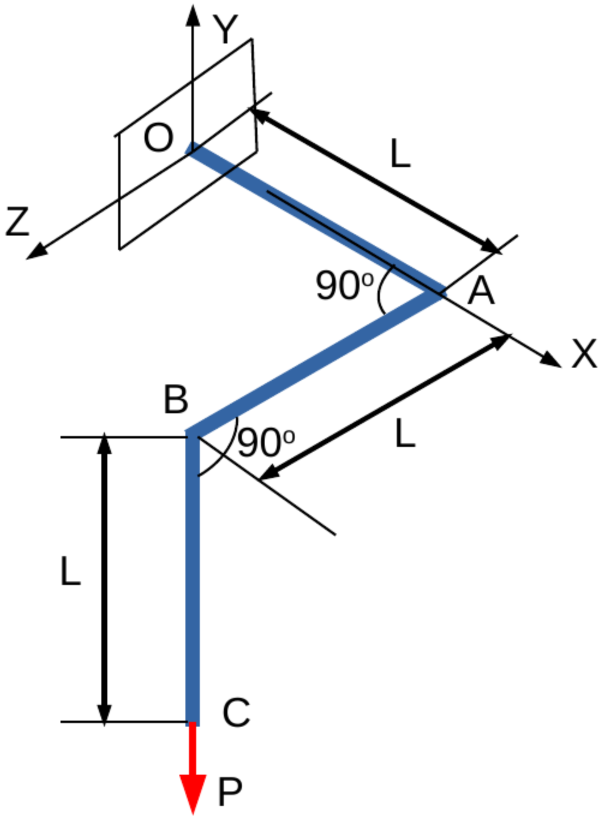

Figure 2: Problem 2

(a) Identify the type of loadings in each of the three segments, viz. axial, bending, torsion and direct shear.
(b) Find the vertical deflection of point C due to the vertical force neglecting the effect of direct shear.
(c) Find the horizontal deflection in the $z-$direction at point C due to the vertical force neglecting the effect of direct shear.

B3. Consider the arrangement of two massless elastic beams, AB and CD, as shown in the figure. (**10 marks**)

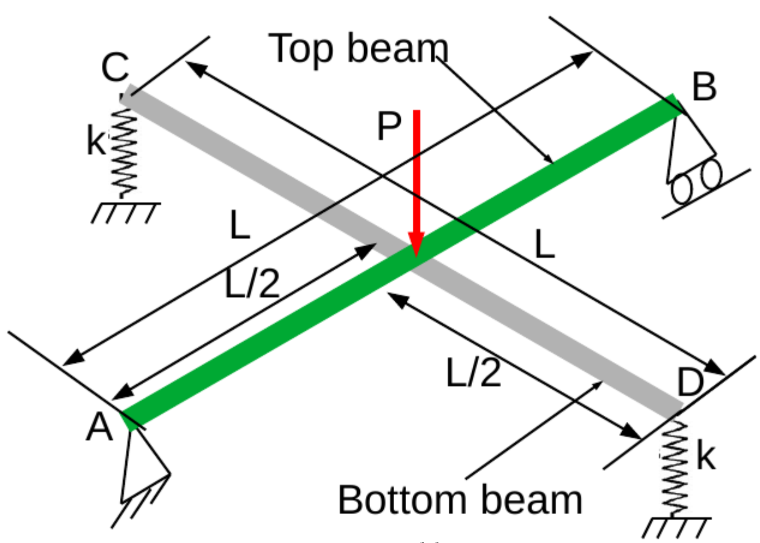

Figure 3: Problem 3

Each beam has length L and a circular cross-section with radius r. The top beam, AB, has Young's modulus E_T while the bottom beam, CD, has Young's modulus E_B. The top beam is simply supported while the bottom beam rests on two linear springs with spring stiffness k. Initially the two beams are just in contact with each other. A concentrated force P acts on the midpoint of the top beam in the vertical direction.

(a) Find ratio of the maximum bending stress in the top beam to the maximum bending stress in the bottom beam due to the applied load P.
(b) Plot the result obtained in part (a) as a function of k if $E_T/E_B = 1$.
(c) Comment on the results obtained in (b) if $k \to \infty$.