Indian Institute of Technology Bombay
Department of Mechanical Engineering
Design 1 - PhD Qualifier Exam (July 2022)

Instructions:
1. There is a total of 6 questions in this paper with 5 pages. Marks for each question are as indicated against the question.
2. You can attempt all the questions. **Solve each part on a separate answer booklet.**
3. Maximum marks: 100. Minimum passing marks: 40.
4. The examination is close books and close notes of any kind. A set of equation that may help solve the paper are given at the end.
5. State your assumptions clearly. Clearly write the formulae you are using during the solution.

Part A

(Solve in a separate answer-book.)

1. **[20 points]** Please solve these two problems:

(a) **[8 points]** In the absence of any body forces, consider a kinematically infinitesimal stress field whose matrix of scalar components in the vector basis $\{\hat{e}_i\}$ is

$$\boldsymbol{\sigma} = \begin{bmatrix} x_2^2 + k(x_1^2 - x_2^2) & -2kx_1x_2 & 0 \\ -2kx_1x_2 & x_1^2 + k(x_2^2 - x_1^2) & 0 \\ 0 & 0 & k(x_1^2 + x_2^2) \end{bmatrix}$$

1. Determine whether the body is at rest. **[1 point]**
2. Determine the traction vector acting at point $\boldsymbol{x} = \hat{e}_1 + 2\hat{e}_2 + 3\hat{e}_3$ on the plane $3x_1 + 4x_2 + 5x_3 = 6$. **[2 points]**
3. Determine the normal and projected shear tractions acting at this point on this plane. **[2 points]**
4. Determine the principal stresses and principal directions of stress at this point. **[3 points]**

(b) **[12 points]** A 2D rectangular beam $-a < x < a, -b < y < b$ is loaded by a uniform compressive normal traction p on $y = b$, zero normal traction on $y = -b$ and simply supported at the ends as shown in Fig. 1.

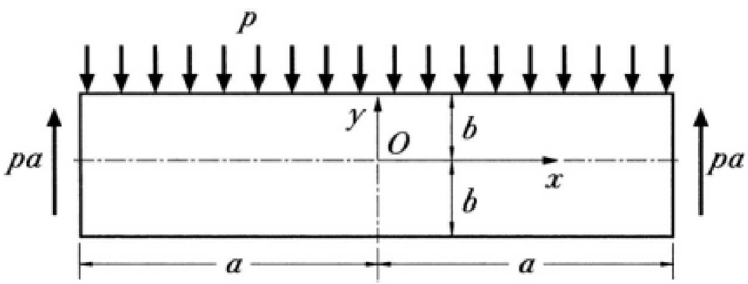

Figure 1: A rectangular beam

1. Write down the stress boundary conditions on $y = \pm b$. Then use symmetry and write down the force and moment equilibrium conditions on $x = a$ to reflect that the beam is simply supported at $x = a$. **[5 points]**

2. Take 5th order Airy stress function, i.e $\phi = C_1x^2 + C_2xy + c_3y^2 + C_4x^3 + C_5x^2y + C_6xy^2 + C_7y^3 + C_8x^4 + C_9x^3y + C_{10}x^2y^2 + C_{11}xy^3 + C_{12}y^4 + C_{13}x^5 + C_{14}x^4y + C_{15}x^3y^2 + C_{16}x^2y^3 + C_{17}xy^4 + C_{18}y^5$. We know that Airy stress function must satisfy the biharmonic equation

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0, \tag{1}$$

and the stress components are given by

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}, \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}. \tag{2}$$

Given these information, write down a linear system of equations through which 18 constants C_is, $i = 1, \ldots, 18$ can be determined. You do not have to calculate the values of C_is. [**7 points**]

2. [**13 points**] The composite beam AB of Fig. 2 consists of two segments with flexural rigidies EI and μEI and lengths L and λL, respectively. The beam is fixed at the bottom and free at the top and is being compressed by an axial force (load) P. Find the transcedental equation determining the critical value of P for buckling instability. Please note that you do not have to find the exact expression for the critical load. You can take the vertical axis as the x-axis and the horizontal axis as the y-axis.

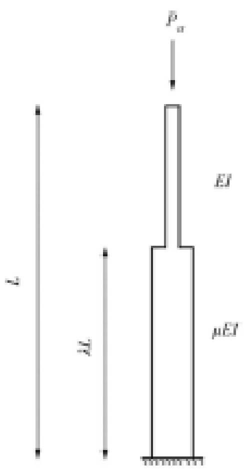

Figure 2: A cantilever composite beam with two different cross-sections, length and flexural regidities

PART B

3. Consider a linear elastic, isotropic, Euler-Bernoulli beam of length L subjected to loading as shown in Figure 1 below. The modulus of rigidity of the beam is EI.

The total potential energy Π of an elastic system is the sum of the stored strain energy and the potential energy of the external forces. The Principle of minimum potential energy states that the static equilibrium of an elastic body implies that the total potential energy Π must be minimum with regards to any kinematically admissible small variation in the displacement field. Here, kinematically admissible displacement field means a displacement field which is single-valued, continuous and satisfies the displacement boundary conditions of the problem under consideration.

(a) Write down the expression for Π for the beam shown in Figure 1. [**4 Points**]

(b) Check if $v(x) = a_o \sin\left(\frac{\pi x}{L}\right)$, with a_o being an undetermined constant, is a kinematically admissible displacement field for the beam shown in Figure 1? If yes/no, why? [**2 Points**]

(c) Use $v(x)$ from (b) and compute an expression for Π. [**3 Points**]

(d) Minimize Π with respect to a_o, apply Principle of minimum potential energy and compute the expression for the constant a_o. [**5 Points**]

(e) Determine the expression for the elastic curve or deflection of the beam $v(x)$ in terms of $P_{cr} = \frac{\pi^2 EI}{L^2}$. Comment on the physics of the deformation of the beam as $P \rightarrow P_{cr}$. [**2.5 Points**]

[Information: The axial shortening of a beam due to lateral deflection v(x) is given by:

$$\Delta_v = \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx.]$$

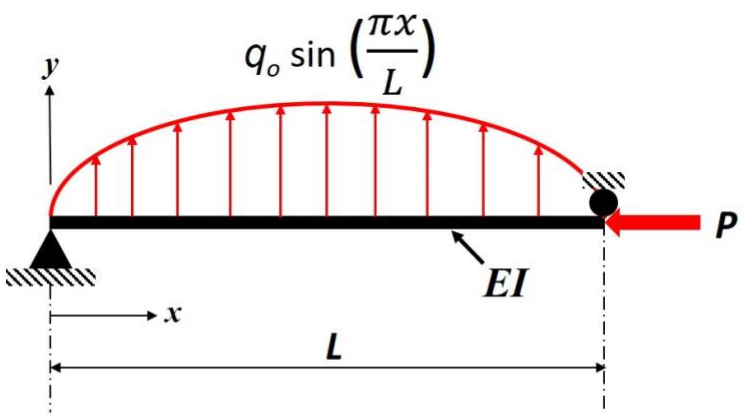

Figure 1

4. Given the following system of strains:

$$\varepsilon_{xx} = 5 + x^2 + y^2 + x^4 + y^4$$
$$\varepsilon_{yy} = 6 + 3x^2 + 3y^2 + x^4 + y^4$$
$$\gamma_{xy} = 10 + 4xy(x^2 + y^2 + 2)$$
$$\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0.$$

(a) What state of strain does the above system represent in general? [**1 Point**]

(b) Verify whether the above strain field is possible or not. [**2 Points**]

(c) If the above strain field is possible, determine the displacement components u_x and u_y in terms of x and y, assuming that $u_x = u_y = 0$ along with the rotation about the z-axis $\omega_{xy} = 0$ at the origin. [**7.5 Points**]

(d) Determine the maximum, intermediate and minimum principal strains at the origin. [**3 Points**]

(e) Determine the orientations of the maximum, intermediate and minimum principal strain directions at the origin. [**3 Points**]

Part-C

5(a). [10 Marks] Establish the relationship between elastic constants $E, G\ and\ \nu$ for a linear elastic, homogeneous and isotropic material using any two-dimensional (2D) boundary value problem of your choice. The solution to the problem in terms of stress and strain should lead to the relationship. Make sure you write down all your assumptions clearly with justification.

5(b). [5+5 = 10 Marks] Consider a very large thin, linear, elastic, isotropic and homogeneous plate with Young's modulus, E, Poisson's ratio, ν. The plate has a hole of radius 'a' at its center and is subjected to an in-plane equibiaxial stress state far from the hole. (a) Setup the boundary value problem clearly with all the governing equations and boundary conditions accompanied with a schematic, and (b) find the stress concentration on the periphery of the hole using the tangential stress component. Make sure you write down all your assumptions clearly with justification.

6. [7+7 = 14 Marks] Consider a linear, elastic, and homogeneous solid cylindrical shaft of radius 'a' and length 'L'. The shaft is fixed at one end and is subjected to a torque 'T' all along its length. If the shear modulus of the shaft is varying exponential along the radius as $G(r) = G_0 e^{rK}$, where $G_0 > 0\ and\ K > 0$ are constants. Without using the torsion formula including any of its alternate or derivative expressions, and using only the kinematic, equilibrium and constitutive relationships, find (a) Stress and (b) Strain, fields at every point in the shaft. Make sure you list out all your assumptions upfront with appropriate justification.

USEFUL RELATIONS IN ELASTICITY

DISCLAIMER: STANDARD NOTATION OF OPERATORS AND SYMBOLS APPLIES.

Equilibrium Equations: Cartesian coordinates:

(1) $$\sigma_{ij,j} + b_i = 0$$

Cylindrical coordinates:

(2) $$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) + b_r = 0\\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta}}{\partial \theta\theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + b_\theta = 0\\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + b_z = 0
\end{aligned}$$

Strain-Displacement relation: Cartesian coordinates:

(3) $$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Cylindrical coordinates:

(4) $$\begin{aligned}
&\varepsilon_{rr} = \frac{\partial u_r}{\partial r},\ \varepsilon_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right),\ \varepsilon_{zz} = \frac{\partial u_z}{\partial z}\\
&\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\\
&\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)\\
&\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)
\end{aligned}$$

Saint Venant compatibility equations:

(5) $$\begin{aligned}
&\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}\\
&\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}\\
&\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x}\\
&\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right)\\
&\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right)\\
&\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right)
\end{aligned}$$

Hooke's law:

(6)
$$\begin{aligned}
\sigma_{ij} &= \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\,\alpha\,(T - T_0)\,\delta_{ij}\\
&= \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right) - \frac{E}{1-2\nu}\alpha\,(T - T_0)\,\delta_{ij}
\end{aligned}$$