Sample Paper: Math Comprehensive Examination

Day: XXX — Max Time: 3 hours
Date: XXX — Max Marks: 60
Passing Marks: 30

Note: This is an open notes/books exam. There are **SEVEN** questions.

1. Consider the following system of first-order ODEs: (**10 marks**)

$$t\frac{d\mathbf{y}(t)}{dt} + \mathbf{A}\mathbf{y} = \mathbf{b}$$

where

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

   (a) Find the eigenvalues λ_1,λ_2 and respective eigenvectors $\mathbf{e}_1$,$\mathbf{e}_2$ of $\mathbf{A}$.

   (b) Assuming the form $\mathbf{y}(t) = \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2$, and $\mathbf{b} = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$, obtain uncoupled ODEs for $\alpha_1(t)$ and $\alpha_2(t)$.

   (c) Using the initial conditions $y_1(1) = y_2(1) = 1$, find the solutions $y_1(t)$ and $y_2(t)$ for $t > 0$.

2. The temperature in a body is given by $T = x^2 + xy + yz$. At the point (2,1,4) what is the value of the derivative of the temperature (i) in the x direction, and (ii) in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$? Also determine the direction in which the temperature is changing most rapidly at the point (2,1,4). (**5 marks**)

3. Evaluate $\int_A \underline{\mathrm{u}} \cdot \hat{n}dA$ in two ways, if $\underline{\mathrm{u}} = x\hat{i} + x\hat{j} + z^2\hat{k}$ and A is the cylinder bounded by $z = 0$, $z = 8$ and $x^2 + y^2 = 4$. (**5 marks**)

4. Use the method of separation of variables to obtain the solution $u(x,t)$ of the partial differential equation (**10 marks**)

$$u_t = 2u_{xx} - u_x, \quad 0 < x < 1$$

   subject to the boundary conditions $u(0,t) = 0, u(1,t) = 0$ and initial condition $u(x,0) = 3e^{x/4}\sin(5\pi x)$.

5. Use the appropriate Fourier transform to obtain the solution $u(x,t)$ of the initial value problem (Cauchy problem) (**10 marks**)

$$\begin{aligned} u_t &= u_{xx} + 2u + (1 - 4x^2 t)e^{-x^2}, \quad -\infty < x < \infty, t > 0 \\ u(x,t), u_x(x,t) &\to 0 \quad \text{as} \quad x \to \pm\infty, t > 0 \\ u(x,0) &= 0, \quad -\infty < x < \infty \end{aligned}$$

6. When solving the problem of a vibrating string with variable density using the separation variations technique, one encounters the following eigenvalue problem (**10 marks**)

$$\frac{d^2X}{dx^2} + \frac{\lambda}{(1+x)^2}X = 0, \;\; 0 < x < 1,$$

with $X(0) = 0$ and $X(1) = 0$. Find the eigenvalues and the corresponding eigenfunctions.

7. Consider a case with an undamped spring mass system is subjected to different forcing functions in different time intervals. The governing equation of motion of such a system is given by **(10 marks)**

$$\frac{d^2x}{dt^2} + x = f(t),$$

where

$$f(t) = \begin{cases} t, & 0 \leq t \leq \pi, \\ \pi e^{\pi - t}, & t > \pi. \end{cases}$$

The initial conditions are $x(0) = 0$ and $\frac{dx}{dt}(0) = 1$. Find the displacement $x(t)$. Note that one needs to enforce the requirement that the displacement and the velocity is continuous at all time.