PhD Qualifying Exam 20th January 2020: [Max 60pts]: Time: 3 hours

- Closed book. Closed notes. Calculators permitted.
- If you think any data or information is missing, make and underline a reasonable assumption.
- Invigilators will not clarify any doubts.
- Solve each question on a new page. To pass one needs to obtain 24 points.

1. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix}$
   (a) Show that A is diagonalizable. [2 pts]
   (b) Find the eigenvalues of A. [2 pts]
   (c) Find an orthogonal basis of eigenvectors of A with respect to the usual inner product. [3 pts]

2. Consider a system of differential equations given below.
$$\begin{cases} x_1'(t) = 3x_1(t) + x_2(t) + x_3(t) \\ x_2'(t) = 2x_1(t) + 4x_2(t) + 2x_3(t) \\ x_3'(t) = -x_1(t) - x_2(t) + x_3(t) \end{cases}$$
   where $x_i(t)$ is real-valued differentiable function of real variable t for all i. Determine all the solutions of the system of differential equations. [8 pts]

3. There is a rabbit that runs on the circumference of a perfect circle of radius R with a constant speed v. A fox chases the rabbit, starting from the center of the circle and also moves with a constant speed v such that it is always between the center of the circle and the rabbit. How long will it take for the fox to catch the rabbit? [8 pts]

4. A rocket that Pintu bought last Diwali did not go straight up in the sky. It took a weird curvy path and then to a neighbor's house. Pintu has been looking for return of money from his vendor. Vendor has agreed to return the money if Pintu could prove that $\kappa = \tau$ for the curvy path of the rocket. Help Pintu get his money back from the vendor. Pintu has estimated a curve based on the multiple CCTV cameras footage along the street and it is $x = t, y = t^2, z = \frac{2}{3}t^3$. κ and τ are curvature and torsion of the curve respectively. [8 pts]

5. During last Mood Indigo, Pintu won a skateboard in an event and has been trying to learn skateboarding. Let us help him. Before he could do anything about sliding his skateboard, he would like to know the tangent plane on the surface where he wants to slide. What would be the tangent plane on surface $z = 3x^2 + 2y^3$ at the point $(1, -1, 2)$? [7 pts]

6. For $\vec{F} = y\hat{i} + (x - 2xz)\,\hat{j} - xy\hat{k}$, evaluate

$$\iint_R \left(\nabla \times \vec{F}\right).\vec{n}dS$$

where $S = x^2 + y^2 + z^2 = a^2$ for $z \geq 0$. [7 pts]

7. Using the method of undetermined coefficients find the general solution to the following differential equation. [8 pts]

$$y''' - y'' - y' + y = 2e^{-t} + 3$$

8. Find the general solution to the following second order ODE. [7 pts]

$$y'' + y = \frac{1}{sin(x)}$$
