Department of Mechanical Engineering, IIT Bombay

PhD Qualifier Exam: Applied Mathematics: 23 Jan 2019

Max. Marks: 60 Duration: **3 Hours** **Min. passing marks: 24**

Closed notes / book exam; A list of common laplace transform pairs enclosed

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1. (5=2+3 points) Let W be the three dimensional region under the graph of $f(x,y) = e^{x^2+y^2}$ and over the region in the plane defined by $1 \leq x^2 + y^2 \leq 2$.

   (a) Find the volume of W.

   (b) Find the flux of the vector field $F = (2x - xy)\hat{\imath} - y\hat{\jmath} + yz\hat{k}$ out of the region W.

2. (7=1+2+2+2 points) Let C be the curve $x^2 + y^2 = 1$ lying in the plane $z = 1$. Let $F = (z - y)\hat{\imath} + y\hat{k}$.

   (a) Calculate $\nabla \times F$.

   (b) Calculate $\int_C F \cdot ds$ using a parametrization of C and a chosen orientation of C.

   (c) Write $C = \partial S$ for a suitably chosen surface S and , applying Stokes' theorem. Verify your answer in (b).

   (d) Consider the sphere with radius $\sqrt{2}$ with it's center at the origin. Let S' be the part of the sphere that is above the curve (i.e., lies in the region $z \geq 1$), and has C as boundary. Evaluate the surface integral of $\nabla \times F$ over S'. Specify the orientation you are using for S' by drawing a figure.

3. (8 points) For the 2^{nd} order ODE, $(x^2 - 1)y'' - 2xy' + 2y = 0$, $y = x$ is the first solution. Find the second independent solution and verify by substitution.

4. (8 points) Consider a system of differential equations given below.

$$\begin{cases} x_1'(t) = 3x_1(t) + x_2(t) + x_3(t) \\ x_2'(t) = 2x_1(t) + 4x_2(t) + 2x_3(t) \\ x_3'(t) = -x_1(t) - x_2(t) + x_3(t) \end{cases}$$

   where $x_i(t)$ is real-valued differentiable function of real variable t for all i. Determine all the solutions of the system of differential equations.

5. (7=6+1 points) Consider the following solution to one-dimensional wave equation:
   $y = A\cos(\omega t)\sin(kx)$

   (a) Determine functions $f(t - x/c)$ and $g(t + x/c)$ such that their sum is equal to the expression given above. Here $k = \omega/c$.

(b) What is the physical significance of the two functions determined in (a)?

6. (7 points) The governing equation of a resistor-capacitor circuit subjected to a voltage $V(t)$ is given by $R\dot{q}+\frac{1}{C}q = V(t)$ where q is the electrical charge. Using Laplace transform method, determine the response of the circuit to a unit voltage impulse at t = 0. Refer to the table at the end for additional information.

7. (6 points) Find the Fourier series for the impulse train shown in the figure below

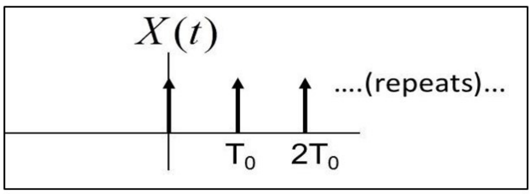

Figure for question 7

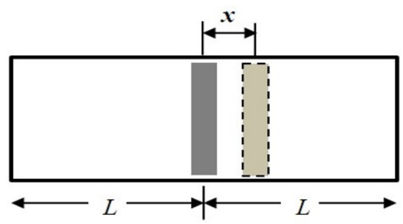

Figure for question 8

8. (12 = 4+4+4 points) In a piston oscillator, a piston of mass m is placed at the mid-point of a closed cylinder of cross-sectional area A and legth $2L$ as shown in the figure (above). Assume that pressure p on either side of the piston satisfies $PV = constant$ (Boyle's law). Neglect the friction at the cylinder-piston contact surface. Let P_0 be the pressure on both sides when $x = 0$.

(a) If the piston is distrubed from its equilibrium position $x = 0$, show that the governing equation of motion is

$$m\ddot{x} + 2P_0AL(\frac{x}{L^2 - x^2}) = 0$$

(b) Use Taylor series to expand the term $x/(L^2-x^2)$ about $x = 0$ (equilibrium position) and retain only the leading term to derive the linearized version of the equation of motion: $m\ddot{x} + 2P_0Ax/L = 0$

(c) Determine the frequency of oscillation of the linearized system.