Department of Mechanical Engineering, IIT Bombay

Ph.D. Qualifying Examination: Fluid Mechanics (TFE-1)

Maximum marks: 75 **24th January, 2020** **Time: 2:30 PM-5:30 PM**

This is a **closed book, closed notes** examination. ***Use of calculator is allowed.*** Answer all questions. Make suitable assumptions if required and state them clearly. Simplify the answers to the extent possible. Useful equations are given in the Appendix. Please write legibly. *Best of luck for the exam!*

Problem 1A: Fundamentals of Theoretical Fluid Dynamics [14 Marks]:

Consider a fluid (density ρ) flowing through an infinitesimal 2D Cartesian control volume (CV) of size δx and δy along x and y-axis, respectively.

a) **[4+4 marks]** Consider velocity components as u and v at northwest corner of the CV, respectively. Derive an expression for the linear strain rates in x and y directions using Taylor series. Show that the volumetric strain rate of the particle is equal to the divergence of velocity vector.

b) **[6 marks]** Consider state of stress at O (centre of particle) as $\{\sigma_{xx}, \sigma_{yy}, \tau_{xy}, \tau_{yx}\}$, where symbols and subscripts have usual meanings. Show that the stress tensor is symmetric for this system.

Problem 1B: **Fundamentals of Theoretical Fluid Dynamics [6 marks]**

Consider the free stream flow (velocity U_∞) over a surface (length L) with x axis along the surface and y axis being normal to the surface, with p_∞ as ambient pressure. Obtain non-dimensional form of y-component of the incompressible Navier-Stokes equation given below. Identify three dimensionless numbers that you obtain after non-dimensionalization.

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

Problem 2: Kinematics of Fluid Flow [10 Marks]

a) **[2 marks]** For a 2D Cartesian coordinate system, present the expressions for (i) Lagrangian and (ii) Eulerian description of acceleration of a fluid particle – in terms of velocity gradients.

b) **[2 marks]** For a steady flow, is it possible for a fluid particle to experience non-zero acceleration? Discuss it with the help of mathematical expressions and an example problem on flow in a nozzle.

c) **[3+3 marks]** Using the fluid kinematics in 2D Cartesian coordinate system, derive the expressions for (i) shear strain rate and (ii) rate of rotation vector; as a function of velocity gradient.

Problem 3: Boundary layer [10 marks]

Starting from the Navier-Stokes equations derive the equations for laminar boundary layer over a flat plate. You can assume that the pressure gradient along streamwise direction is zero. All the steps and arguments involved in the derivation should be clearly given.

Problem 4: Exact Solutions [15 marks]

Consider a large plate surrounding by fluid (the fluid extends to infinity). The plate is impulsively set in motion in its plane (*x*-*z* plane) with a constant velocity *U*. The interest is in examining the resulting flow on one side of the plate, i.e., $u(y, t)$. This problem admits a similarity solution. You are required to find the solution of the problem, with the help of some information given below.

Note that there is a single variable – velocity component along the length of the plate (u). The governing equation is:

$$\frac{\partial u}{\partial t} = v \frac{\partial^2 u}{\partial y^2}$$

Similarity allows us to write: $u/U = F(\varphi)$ where $\varphi = y/(2\sqrt{vt})$.

a) **[3 marks]** Draw a sketch and write the initial and boundary conditions for the problem.

b) **[4 marks]** Transform the governing equation from partial to ordinary differential equation with the help of the above similarity variable.

c) **[5 marks]** Solve the resulting ordinary differential equation.

d) **[3 marks]** Complete the solution by evaluating the constants of integration.

Problem 5: Potential Flow [8 marks]

Consider 2D inviscid flow near a right angle corner where the velocity can be modeled as a steady flow given by $\mathbf{u} = Ax\hat{\mathbf{e}}_x + Ay\hat{\mathbf{e}}_y$ where *A* is a positive constant.

a) **[2 marks]** In figure 1a by examining the curvature of the streamlines indicate the regions of high and low pressure. Also show the direction of acceleration.

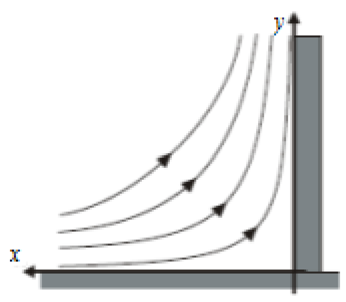

Fig. 1a

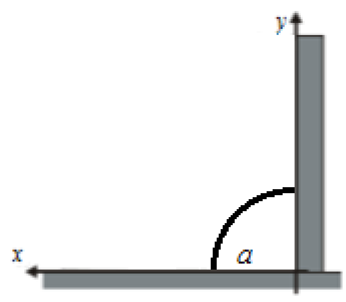

Fig. 1b

b) **[6 marks]** Now suppose that a circular cylinder of radius *a* is introduced with its axis coincident with the corner as shown in Fig. 1b. Sketch the streamlines for this case. Assuming that the resultant flow is a potential flow, solve for the stream function and the velocity components. [*Hint:* You may find it easier to work in polar co-ordinates].

Problem 6: Vortex Dynamics [12 marks]

Consider a Rankine vortex defined by the azimuthal component of velocity

$$u_\theta(r) = \begin{cases} \Gamma r / 2\pi\sigma^2, & for\ 0 < r \le \sigma \\ \Gamma / 2\pi r, & for\ r > \sigma \end{cases} \text{ where } r \le \sigma \text{ is the core of the vortex.}$$

a) **[4 marks]** For this vortex, one pair of streamlines (in black) is situated at $r < \sigma$ and another pair (grey) at $r > \sigma$ as shown in figure 2. Also shown are fluid elements at t = 0: ABCD bounded by the black streamlines and PQRS bounded by the grey streamlines. If both the elements remain bounded by the streamlines, indicate their orientation and shape at any time t > 0 assuming they remain in the first quadrant. Provide reasons for your answer.

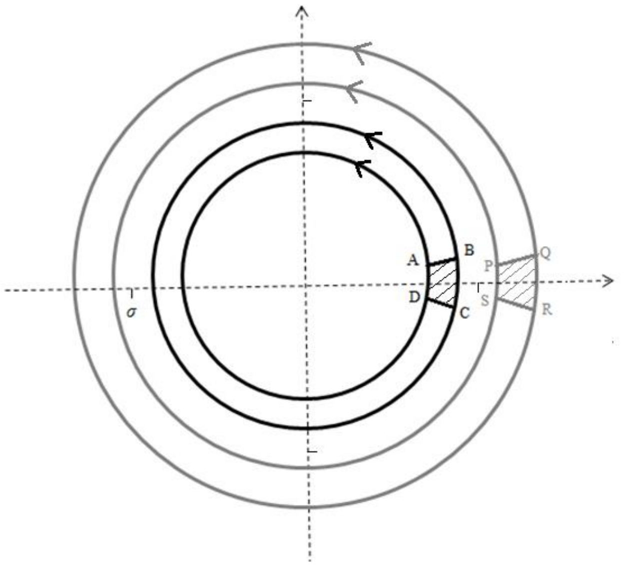

Fig. 2

b) **[4 marks]** Identify the region in the vortex where the net viscous force is zero even though viscous stresses are present. Show your calculations. How is this possible (since force is stress times area)?

c) **[4 marks]** A tornado can be idealized as a Rankine vortex with a core of diameter 30 m. The gauge pressure at a radius of 15 m is −2000 N/m^2 (that is, the absolute pressure is 2000 N/m^2 below atmospheric).

(i) Calculate the circulation around a closed loop of any shape surrounding the core.

(ii) Assume that the tornado is moving at a linear speed of 15 m/s relative to the ground. Find the time required for the gauge pressure to drop from −500 to −2000 N/m^2. Neglect compressibility effects and assume the air density to be 1.2 kg/m^3.

Plane Polar Coordinates

Position and velocity vectors $\mathbf{x} = (r,\, \theta) = r\mathbf{e}_r$; $\mathbf{u} = (u_r,\, u_\theta) = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta$

Gradient of a scalar ψ: $\nabla\psi = \mathbf{e}_r\dfrac{\partial\psi}{\partial r} + \mathbf{e}_\theta\dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$

Laplacian of a scalar ψ: $\nabla^2\psi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\psi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\psi}{\partial\theta^2}$

Divergence of a vector: $\nabla\cdot\mathbf{u} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(ru_r) + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}$

Curl of a vector, vorticity: $\boldsymbol{\omega} = \nabla\times\mathbf{u} = \mathbf{e}_z\left(\dfrac{1}{r}\dfrac{\partial(ru_\theta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}\right)$

Laplacian of a vector: $\nabla^2\mathbf{u} = \mathbf{e}_r\left(\nabla^2 u_r - \dfrac{u_r}{r^2} - \dfrac{2}{r^2}\dfrac{\partial u_\theta}{\partial\theta}\right) + \mathbf{e}_\theta\left(\nabla^2 u_\theta + \dfrac{2}{r^2}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta}{r^2}\right)$

Strain rate S_{ij} and viscous stress σ_{ij} for an incompressible fluid where $\sigma_{ij} = 2\mu S_{ij}$:

$S_{rr} = \dfrac{\partial u_r}{\partial r} = \dfrac{1}{2\mu}\sigma_{rr},\, S_{\theta\theta} = \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r} = \dfrac{1}{2\mu}\sigma_{\theta\theta},\, S_{r\theta} = \dfrac{r}{2}\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right) + \dfrac{1}{2r}\dfrac{\partial u_r}{\partial\theta} = \dfrac{1}{2\mu}\sigma_{r\theta}$

Equation of continuity: $\dfrac{\partial\rho}{\partial t} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\rho u_r) + \dfrac{1}{r}\dfrac{\partial}{\partial\theta}(\rho u_\theta) = 0$

Navier-Stokes equations with constant ρ, constant ν, and no body force:

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right),$$

where $\nabla^2 = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$.