Quiz#1

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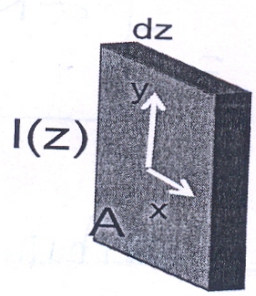
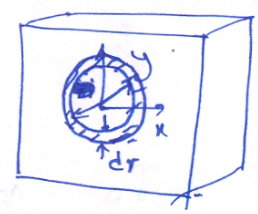
Instructions:
You could use your notes.
Invigilators can award zero if any unfair means is resorted.

1. A functionally graded material does not have uniform distribution of opaque particles instead there are more number of opaque particles in the center and reduce significantly towards outer edges. The volumetric density (particles/volume) is a function of radial distance from the center and could be modeled as $N(r) = N_0 e^{\left(\frac{-2r^2}{r_i^2}\right)}$. Where, N_0 is the maximum volumetric density (at the center), $r = \sqrt{x^2+y^2}$, and r_i is the radius of the inscribed circle. Assume x and y vary from –a to +a. The origin is located at the center of the plate. Derive an expression for Beer Lambert's law of absorption for such material.

<u>Assume that outside the inscribed circle there are few particles and they could be ignored. Each particle has same area, σ.</u>

[20]





Fraction of photons absorbed $= \dfrac{\text{Opaque area}}{\text{Total area}}$

Opaque area [annulus] $= \sigma\, N(r)(dA) \times (dz)$

$$= \sigma N_0 e^{\left(-\frac{2r^2}{r_i^2}\right)} \times dA \cdot dz$$

$$= \sigma N_0 e^{-\frac{2(x^2+y^2)}{r_i^2}} \times dx\,dy\, dz.$$

∴ Total opaque area $= \left[\iint_A \sigma N_0 e^{-\frac{2(x^2+y^2)}{r_i^2}} dx\,dy\right] dz$

$$= \left[\iint_A \sigma N_0 e^{-\frac{2r^2}{r_i^2}} \times 2\pi r\, dr\right] dz$$

$\therefore$ Fractions of photons absorbed

$$= \frac{\text{Opaque area}}{\text{Total area}}$$

$$\boxed{\frac{dI}{I} = \frac{-\left[\sigma N_0 \iint_A e^{-\frac{2r^2}{r_i^2}} \times 2\pi r\, dr\right] dz}{a^2}} \quad \text{--- (1)}$$

It can be transformed to cartesian co-ordinates as well

i.e $\frac{dI}{I} = \frac{-\left[\sigma N_0 \iint_A e^{-\frac{2(x^2+y^2)}{r_i^2}} \times dx\, dy\right] dz}{a^2}.$

To get the distribution of intensity

$$\boxed{\int_{I_0}^{I} \frac{dI}{I} = -\int_0^z \frac{\left[\sigma N_0 \iint_A e^{-\frac{(2(x^2+y^2))}{r_i^2}}\, dx\, dy\right]}{a^2}} \quad \text{--- (2)}$$

Solving equation (1)

$$\frac{dI}{I} = \frac{-\left[\sigma N_0 \int_0^{r_i} e^{-\frac{2r^2}{r_i^2}} \times 2\pi r \cdot dr\right] dz}{a^2}$$

$$\Rightarrow \frac{dI}{I} = \frac{-\sigma N_0 (-1 + e^{2}] \pi r_i^2\, dz}{2e^2}$$

Integrating the above equation

$$\int_{I_0}^{I} \frac{dI}{I} = \int_{0}^{z} \frac{\sigma N_0 [e^2 - 1]}{2e^2} \pi r_i^2 \, dz$$

$$\Rightarrow \ln\left(\frac{I}{I_0}\right) = -\frac{\sigma N_0 [e^2 - 1]}{2e^2} \pi r_i^2 z$$

$$\Rightarrow \boxed{I(z) = I_0 \, e^{-\frac{\sigma N_0 [e^2 - 1]}{2e^2} \pi r_i^2 z}}$$

$\because$ r_i is the radius of inscribed circle

$$r_i = a.$$

$$\therefore \boxed{I(z) = I_0 \, e^{-\frac{\sigma N_0 (e^2 - 1)}{2e^2} \pi a^2 z}}$$

2. A single focusing lens is required for focusing the collimated beam from laser delivery system. Find the following:
   a) Focal length required for a 2 mm (z_1= 25.4 mm) collimated 930 nm beam to be focused down to 500 µm. Find the prediction error when approximate formula is used. [7.5]
   b) The beam size D(z) at z= 300 mm in the region between the focusing lens and the beam waist. [5]
   c) The depth of focus for the given focusing optics. [7.5]

Assume M^2=1.1.

(a)
→. $D_{01} = 2\text{mm}$, $z_1 = 25.4\text{mm}$, $\lambda = 930\text{nm}$

$D_{02} = 500\ \mu\text{m}$

Using the equation

$$\boxed{\frac{1}{D_{02}^2} = \frac{1}{D_{01}^2}\left[1 - \frac{z_1}{f}\right]^2 + \frac{1}{f^2\Theta_1^2}} \quad - (1)$$

Here $\Theta = M^2 \times \dfrac{4\lambda}{\pi D_{01}}$

$$= 1.1 \times \frac{4 \times 930 \times 10^{-9}}{\pi \times 2 \times 10^{-3}}$$

$$\boxed{\Theta = 6.51 \times 10^{-4}} \quad - (2)$$

Putting (2) in (1) and other given values we have,

$$\frac{1}{(500 \times 10^{-6})^2} = \frac{1}{(2\times10^{-3})^2}\left[1 - \frac{25.4\times10^{-3}}{f}\right]^2 + \frac{1}{f^2(6.51\times$$

$$4\times10^6 = \frac{2.5\times10^5}{f^2}\left[f - 25.4\times10^{-3}\right]^2 + \frac{2.3576}{f^2}$$

$$\Rightarrow 4\times10^6 f^2 = 2.5\times10^5\left[f^2 - 50.8\times10^{-3}f + 6.4516\times10^{-}\right.$$

$$+ 2.3576\times10^6.$$

$$\Rightarrow 4\times10^6 f^2 = 2.5\times10^5 f^2 - 12700f + \cancel{161.25}\ 2.3577\times10$$

$$\Rightarrow 3.75\times10^6 f^2 + 12700f - 2.3577\times10^6 = 0.$$

$\dots 791$ m or $f_2 = -0.794$ m

Approximate solution is given as

$$D_{o2} = \frac{4f \cdot \lambda \cdot M^2}{\pi D_{o1}}$$

$$\Rightarrow 500 \times 10^{-6} = \frac{4 \times f \times 930 \times 10^{-9} \times 1.1}{\pi \times 2 \times 10^{-3}}$$

$$\Rightarrow \boxed{f = 76.77\ cm}$$

∴ Prediction error when approximate sol^n is

$$= 79.1\ cm - 76.77\ cm$$

$$= \underline{\underline{2.33\ cm}}$$

(b)

→ To get $D(z)$ at $z = 300\,mm$ we use the equation.

$$\cancel{D(z)^2}\ D_{z2}^2 = D_{o2}^2 + z^2 \theta^2$$

$$\theta = \frac{4 \lambda M^2}{\pi D_{o2}}$$

$$= \frac{4 \times 930 \times 10^{-9} \times 1.1}{\pi \times 500 \times 10^{-6}}$$

$$\theta = 2.605 \times 10^{-3}$$

∴ $D_{z2}^2 = (500 \times 10^{-6})^2 + z^2 (2.605 \times 10^{-3})^2$

$$\boxed{D_{z2}^2 = 2.5 \times 10^{-7} + 6.786 \times 10^{-6} z^2} \quad -(3)$$

∴ ~~at z = 300 mm~~

~~D_{z2}~~ ~~$D_{z2}^2 = 2.5 \times 10^{-7} + 6.786$~~

The above expression gives the value of D_{z2} after it ...

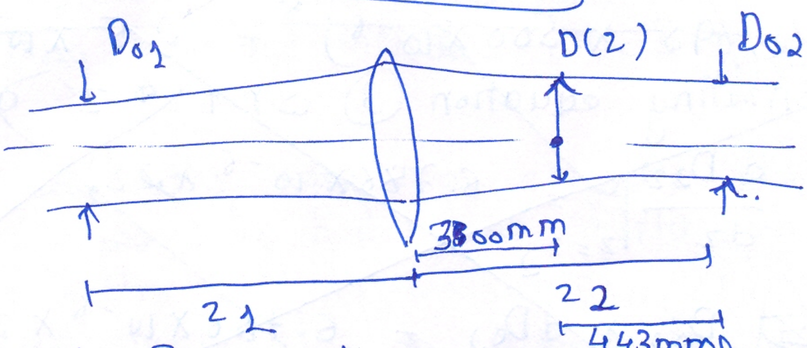
Thus we need to shift the origin.

Using $z_2 = f + \dfrac{(z_1 - f) f^2}{(z_1 - f)^2 + \left(\frac{D_{01}}{\theta_1}\right)^2}$. [Putting $\theta_1 = 6.51 \times 10^{-4}$ from ②]

$$\Rightarrow z_2 = \cancel{79}\ 0.791 + \frac{(25.4 \times 10^{-3} - 0.791) \times (0.791)^2}{(25.4 \times 10^{-3} - 0.791)^2 + \left(\frac{2 \times 10^{-3}}{6.51 \times 10^{-4}}\right)^2}$$

$$= 0.791 - \frac{0.7656 \times 0.6256}{0.5861 + 9.438}$$

$$\boxed{z_2 = 0.7432\,m} \quad -④$$



∴ To get D_{z2} at $z = 300$ mm from lens we put $z = 0.7432 - 0.3$

$$z = 0.443\,m$$

$$\therefore D_{z2}^2 \Big|_{at\ z = 0.443} = 2.5 \times 10^{-7} + 6.786 \times 10^{-6} (0.443)^2$$

$$= 1.58 \times 10^{-6}\,m^2$$

$$= \cancel{1.58\ \mu m}.$$

$$\Rightarrow D_{z2} = 1.257 \times 10^{-3}\,m$$

$$\Rightarrow \boxed{D_{z2} = 1257\ \mu m}$$ at 300mm from lens.

(c)

→. Depth of focus is defined as depth for which the focal spot size changes by ±5%.

Using equation (3)

$$\boxed{D_{2z}^2 = 2.5\times10^{-7} + 6.786\times10^{-6}\,z^2} \quad -(3)$$

Putting $D_{2z} = 1.05\,D_{02}$ and calculating z_1

Illy $D_{2z} = 0.95\,D_{02}$ and calculating z_2

Difference of z_1 & z_2 would give the depth of focus.

$$\Rightarrow (1.05\times500\times10^{-6})^2 = 2.5\times10^{-7} + 6.786\times10^{-6}\,z^2$$

$$\Rightarrow \boxed{z_1 = 0.0614\text{ m}}$$

Illy with $D_{2z} = 0.95\,D_{02}$

$$\Rightarrow \cancel{(0.95\times500\times10^{-6})^2 = 2.5\times10^{-7} + 6.786\times10^{-6}\,z^2}$$

Differentiating equation (3) w.r.t z at $z = z_2$ we have

$$2\,D_{2z}\left.\frac{dD_{2z}}{dz}\right|_{z=z_2} = 6.786\times10^{-6}\times2z_2$$

$$\Rightarrow D_{02}\times dD_{02} = 6.786\times10^{-6}\times z_2\times dz$$

$$\Rightarrow 500\times10^{-6}\times[0.1] = 6.786\times10^{-6}\times0.74321\times dz$$

$$\Rightarrow \boxed{dz = 9.91\text{ m}}$$

[Here $z_2 = 0.7432$ m ~~from~~ (4)

at $dD_{02} = 0.1$]

↑ by defination of Depth of focus

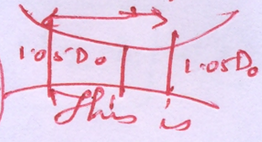
⟹ Putting $D_{2z} = 1.05\,D_{02}$ & calculating z_1

Putting $D_{2z} = 0.95\,D_{02}$ & calculating z_2

Difference of z_1 & z_2 gives depth of focus.

$$\Rightarrow (1.05\,D_{02})^2 = 2.5\times10^{-7} + 6.786\times10^{-6}\,z_1^2.$$

$$\Rightarrow \boxed{z_1 = 0.0614\text{ m}} \longrightarrow \text{Depth of focus is } 2\times0.0614$$



$$\& \cancel{(0.95\,D_{02})^2 = 2.5\times10^{-7} + 6.786\times10^{-6}\,z_2^2}$$

$$\Rightarrow \boxed{z_2 = -0.0599\text{ m}}$$

This is not possible

∴ Depth of focus $= z_1 - z_2$

$= \underline{0.121\text{ m}}$

→ It is minimum