Solution for HW 2

List of abbreviations:

I(x, y) = Heat intensity. (W/mm^2)

P = Laser Power. (W)

σ = Laser beam radius. (mm)

Q = Heat Input. (J)

ρ = Density (Kg/mm^3), a = α = Diffusivity (mm^2/s), K = Thermal conductivity (W/mmk),
c = specific heat capacity (J/kgk).

For the Gaussian heat source, a solution obtained from Green's function for moving point heat source given in Karlslaw and Jaegar can be used. The solution for which satisfies the differential equation of heat conduction in fixed co-ordinate system is [2, 17]:

$$dT'(x, y, z, t') = \frac{2\delta q}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}} \exp[-\frac{(x-vt'-x')^2+(y-y')^2+(z-z')^2}{4a(t-t')}]$$

(3)

In moving coordinate system (see Fig. 2) the moving heat source solution can be written as,

$$dT'(X, Y, Z, t') = \frac{2\delta q}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}} \exp[-\frac{(X-x')^2+(Y-y')^2+(Z-z')^2}{4a(t-t')}]$$

where, δq is the amount of heat released in infinitesimal time increment dt' (δq=Pdt' at instantaneous location (x' + vt', y', z') at time t' shown in Fig. 2. X, Y and Z are moving coordinates (see Fig. 2) where X = x - vt', Y=y and Z=z.

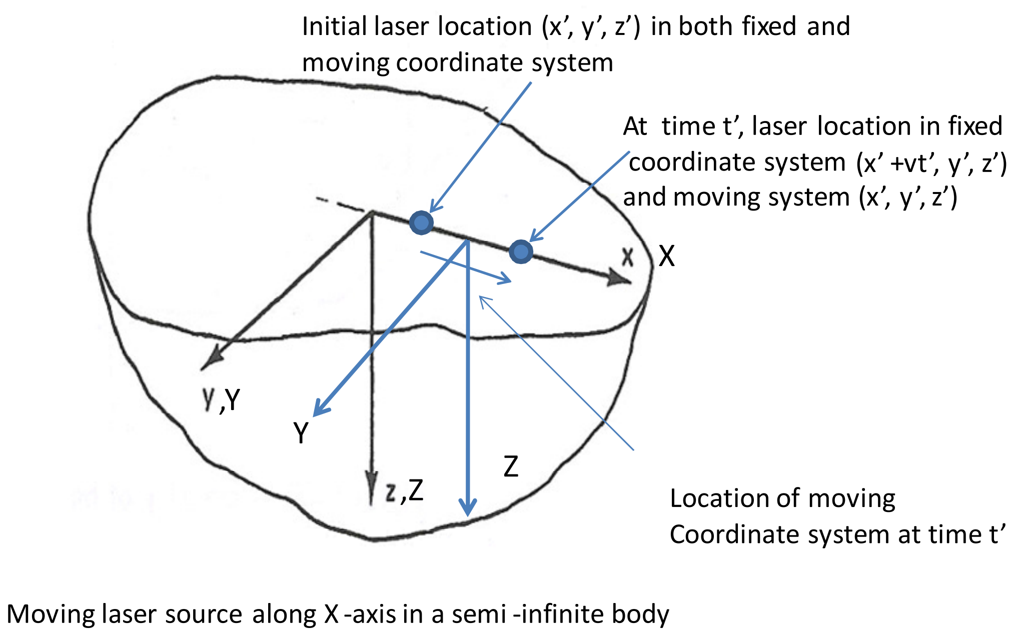

Fig. 2 Schematic diagram of moving point heat source.

The solution of moving Gaussian heat source is the superposition of a series of moving point heat source solutions over the distributed region. By substituting the Gaussian distributed heat source for the point heat source intensity I, this superposition is performed by the integration as shown below.

$$dT' = \frac{2dt'}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}} I(x',y')dx'dy'\exp[-\frac{(X-x')^2+(Y-y')^2+(Z-z')^2}{4a(t-t')}] \quad (4)$$

The laser is incident on surface, therefore z'=0, and the integration of all infinitesimal sources yield,

$$dT'(t') = \frac{4Pdt'}{\pi\sigma^2 \rho C(4\pi a(t-t'))^{\frac{3}{2}}} \times$$

$$\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \exp[-(\frac{2x'^2+2y'^2}{\sigma^2} + \frac{x'^2-2(X)x'+(X)^2+y'^2-2Yy'+Y^2+Z^2}{4a(t-t')})]$$

$$= \frac{4Pdt'}{\pi\sigma^2 \rho C(4\pi a(t-t'))^{\frac{3}{2}}} \frac{\pi\sigma^2 4a(t-t')}{\sigma^2+8a(t-t')} \exp[-\frac{2((X)^2+Y^2)}{\sigma^2+8a(t-t')} - \frac{Z^2}{4a(t-t')}] \quad (5)$$

Rewriting the solution for fixed coordinate system,

$$dT'(t') = \frac{4Pdt'}{\pi\sigma^2 \rho C(4\pi a(t-t'))^{\frac{3}{2}}} \frac{\pi\sigma^2 4a(t-t')}{\sigma^2+8a(t-t')} \exp[-\frac{2((x-vt')^2+y^2)}{\sigma^2+8a(t-t')} - \frac{z^2}{4a(t-t')}]$$

This corresponds to the rise of temperature during a very short time interval from tine t' to t'+ dt' due to an amount of heat Pdt' released on the surface. When considering Gaussian heat source traveling with a constant speed v, the total rise of temperature is sum of all such contributions in the time interval t' = 0 to t' = t.

The summation is carried out by integration is given by,

$$T - T_0 = \frac{4P}{\rho C \pi \sqrt{4a\pi}} \int_{t'=0}^{t'=t} \frac{dt'(t-t')^{-0.5}}{\sigma^2 + 8a(t-t')} \exp[-\frac{2((x-vt')^2 + y^2)}{\sigma^2 + 8a(t-t')} - \frac{z^2}{4a(t-t')}] \quad (7)$$

3.3 Uniform moving circular heat source:

In the Uniform heat source, P is defined as laser power and the radius σ. The heat intensity, I, is given by $\frac{P}{A}$, where A = π*σ^2.

The solution of uniform circular moving heat source is obtained same as Gaussian heat source but only substituting appropriate limits. During a very short time increment dt' the amount of heat released at the surface is δq. This will result in infinitesimal rise in temperature at point (x, y, z) at infinitesimal time dt' given by,

$$dT'(x,y,z,t') = \frac{2dt' I(x',y')dx'dy'}{8\rho C(\pi a(t-t'))^{\frac{3}{2}}} \exp[-\frac{Z^2}{4a(t-t')}] \exp[-\frac{(X-x')^2}{4a(t-t')}] \times \exp[-\frac{(Y-y')^2}{4a(t-t')}] \quad (9)$$

The x' coordinates vary between -σ to +σ and $x^2 + y^2 = \sigma^2$, therefore, the integration of space variables yields,

$$dT(t') = \frac{2Pdt'}{8\rho C \pi \sigma^2 (\pi a(t-t'))^{\frac{3}{2}}} \exp[-\frac{Z^2}{4a(t-t')}] \times \int_{-\sigma}^{\sigma} \exp[-\frac{(X-x')^2}{4a(t-t')}]dx' \int_{-\sqrt{\sigma^2 - x'^2}}^{\sqrt{\sigma^2 - x'^2}} \exp[-\frac{(Y-y')^2}{4a(t-t')}]dy' \quad (10)$$

The total rise in of the temperature can be obtained by integrating from t'=0 to t'= t as, Integrating with respect to space and time variables and transforming back to fixed coordinate system,

$$T - T_0 = \frac{2P}{8\pi^2\sigma^2 K}\int_0^t \frac{dt'}{(t-t')}\exp[-\frac{z^2}{4a(t-t')}]\int_{-\sigma}^{\sigma}\exp[-\frac{((x-vt')-x')^2}{4a(t-t')}]dx'\times$$
$$[-erf(\frac{y-\sqrt{\sigma^2-x'^2}}{2\sqrt{a(t-t')}})+erf(\frac{y+\sqrt{\sigma^2-x'^2}}{2\sqrt{a(t-t')}})] \quad (12)$$

3.4 Uniform moving rectangular heat source:

The rectangular moving heat source has a length and breadth of l and b, respectively. The coordinates lie between –l < x < l and –b < y < b. Heat intensity, I, is constant as mentioned

$$I(x,y) = \frac{P}{A}$$

where A = 4*b*l.

During a very short time increment dt' the amount of heat released at the surface is δqdt'. This will result in infinitesimal rise in temperature at point (x, y, z) at time t given by,

$$= \frac{2\delta q}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\exp[-\frac{Z^2}{4a(t-t')}]\exp[-\frac{(X-x')^2}{4a(t-t')}]\exp[-\frac{(Y-y')2}{4a(t-t')}] \quad (13)$$

$$= \frac{2dt' I(x',y')dx'dy'}{\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\exp[-\frac{Z^2}{4a(t-t')}]\exp[-\frac{(X-x')^2}{4a(t-t')}]\exp[-\frac{(Y-y')2}{4a(t-t')}] \quad (14)$$

$$T - T_0 = \frac{2Pdt'}{4bl\rho C(4\pi a(t-t'))^{\frac{3}{2}}}\exp[-\frac{Z^2}{4a(t-t')}]$$
$$\int_{-l}^{l}\exp[-\frac{(X-x')^2}{4a(t-t')}]dx'\int_{-b}^{b}\exp[-\frac{(Y-y')2}{4a(t-t')}]dy' \quad (15)$$

When considering uniform rectangular heat source traveling on semi-infinite plate with a constant speed v, the total rise of temperature is sum of all such contributions in the time interval t' = 0 to t' = t.

The final solution for moving rectangular heat source is ,

$$T - T_0 = \frac{2P}{8(4bl)\rho C\sqrt{\pi a}} \int_0^t \frac{dt'}{\sqrt{(t-t')}} \exp[-\frac{z2}{4a(t-t')}] \times$$
$$[erf(\frac{x+l+vt'}{\sqrt{4a(t-t')}}) - erf(\frac{x-l+vt'}{\sqrt{4a(t-t')}})][erf(\frac{y+b}{\sqrt{4a(t-t')}}) - erf(\frac{y-b}{\sqrt{4a(t-t')}})] \quad (16)$$

By putting l = b in above equation and we get solution for uniform moving square heat source.

D. Derive the solution for parabolic moving elliptical heat source for a given depth or z:

Intensity distribution for parabolic moving heat source is given by,

The equation is similarly formulated as Gaussian moving heat source only difference is change intensity distribution by parabolic and integrates with appropriate limits. (i.e.) for x it is –a to a and for y it is –b to b.

$$\text{In[1]:=}\ \frac{2}{8\rho C(\pi\alpha(t-t1))^{3/2}}\,\text{Exp}\left[-\frac{(Z-z1)^2}{4\alpha(t-t1)}\right]\int_{-b}^{b}\int_{-a}^{a}\frac{2*P}{a\,b}\left(1-\frac{x1^2}{a^2}\right)\left(1-\frac{y1^2}{b^2}\right)\text{Exp}\left[-\frac{(X-x1)^2+(Y-y1)^2}{4\alpha(t-t1)}\right]dx1\,dy1$$

$$\text{Out[1]=}\ \frac{1}{4C\pi^{3/2}((t-t1)\alpha)^{3/2}\rho}e^{-\frac{(Z-z1)^2}{4(t-t1)\alpha}}\Bigg(-\frac{1}{a^3 b\sqrt{\frac{(a-X)^2}{(t-t1)\alpha}}\sqrt{\frac{(a+X)^2}{(t-t1)\alpha}}}2e^{-\frac{a^2+X^2}{2t\alpha-2t1\alpha}}P\sqrt{\pi}\sqrt{t-t1}\sqrt{\alpha}\Bigg(e^{\frac{a^2+X^2}{2t\alpha-2t1\alpha}}\sqrt{\pi}(a-X)\sqrt{\frac{(a+X)^2}{(t-t1)\alpha}}(X^2+2(t-t1)\alpha)\text{Erf}\left[\frac{1}{2}\sqrt{\frac{(a-X)^2}{(t-t1)\alpha}}\right]+$$

$$\sqrt{\frac{(a-X)^2}{(t-t1)\alpha}}\Bigg(-\frac{2e^{\frac{(a-X)^2}{4(t-t1)\alpha}}(a+X)^2\left(a\left(1+e^{\frac{aX}{t\alpha-t1\alpha}}\right)+\left(-1+e^{\frac{aX}{t\alpha-t1\alpha}}\right)X\right)}{\sqrt{\frac{(a+X)^2}{(t-t1)\alpha}}}+e^{\frac{a^2+X^2}{2t\alpha-2t1\alpha}}\sqrt{\pi}(a+X)(X^2+2(t-t1)\alpha)\text{Erf}\left[\frac{1}{2}\sqrt{\frac{(a+X)^2}{(t-t1)\alpha}}\right]\Bigg)\Bigg)$$

$$\left(\text{Erf}\left[\frac{b-Y}{2\sqrt{t-t1}\sqrt{\alpha}}\right]+\text{Erf}\left[\frac{b+Y}{2\sqrt{t-t1}\sqrt{\alpha}}\right]\right)+\frac{2P\pi(t-t1)\alpha\left(\text{Erf}\left[\frac{a-X}{2\sqrt{t-t1}\sqrt{\alpha}}\right]+\text{Erf}\left[\frac{a+X}{2\sqrt{t-t1}\sqrt{\alpha}}\right]\right)\left(\text{Erf}\left[\frac{b-Y}{2\sqrt{t-t1}\sqrt{\alpha}}\right]+\text{Erf}\left[\frac{b+Y}{2\sqrt{t-t1}\sqrt{\alpha}}\right]\right)}{a\,b}-\frac{1}{a\,b^3}$$

$$2P\sqrt{\pi}\sqrt{t-t1}\sqrt{\alpha}\left(\text{Erf}\left[\frac{a-X}{2\sqrt{t-t1}\sqrt{\alpha}}\right]+\text{Erf}\left[\frac{a+X}{2\sqrt{t-t1}\sqrt{\alpha}}\right]\right)\Bigg(e^{-\frac{b^2+Y^2}{4t\alpha-4t1\alpha}}\Big(-2e^{\frac{bY}{2t\alpha-2t1\alpha}}(t-t1)(b+Y)\alpha+e^{\frac{b^2+Y^2}{4t\alpha-4t1\alpha}}\sqrt{\pi}\sqrt{t-t1}\sqrt{\alpha}(Y^2+2(t-t1)\alpha)\text{Erf}\left[\frac{b-Y}{2\sqrt{t-t1}\sqrt{\alpha}}\right]\Big)+$$

$$e^{-\frac{(b+Y)^2}{4(t-t1)\alpha}}\Big(-2(t-t1)(b-Y)\alpha+e^{\frac{(b+Y)^2}{4(t-t1)\alpha}}\sqrt{\pi}\sqrt{t-t1}\sqrt{\alpha}(Y^2+2(t-t1)\alpha)\text{Erf}\left[\frac{b+Y}{2\sqrt{t-t1}\sqrt{\alpha}}\right]\Big)\Bigg)+$$

$$\frac{1}{a^3b^3\sqrt{\frac{(a-X)^2}{(t-t1)\alpha}}\sqrt{\frac{(a+X)^2}{(t-t1)\alpha}}}2P\Bigg(e^{\frac{a^2+X^2}{2t\alpha-2t1\alpha}}\sqrt{\pi}(a-X)\sqrt{\frac{(a+X)^2}{(t-t1)\alpha}}(X^2+2(t-t1)\alpha)\text{Erf}\left[\frac{1}{2}\sqrt{\frac{(a-X)^2}{(t-t1)\alpha}}\right]+$$

$$\sqrt{\frac{(a-X)^2}{(t-t1)\alpha}}\Bigg(-\frac{2e^{\frac{(a-X)^2}{4(t-t1)\alpha}}(a+X)^2\left(a\left(1+e^{\frac{aX}{t\alpha-t1\alpha}}\right)+\left(-1+e^{\frac{aX}{t\alpha-t1\alpha}}\right)X\right)}{\sqrt{\frac{(a+X)^2}{(t-t1)\alpha}}}+e^{\frac{a^2+X^2}{2t\alpha-2t1\alpha}}\sqrt{\pi}(a+X)(X^2+2(t-t1)\alpha)\text{Erf}\left[\frac{1}{2}\sqrt{\frac{(a+X)^2}{(t-t1)\alpha}}\right]\Bigg)\Bigg)$$

$$\Bigg(e^{-\frac{2a^2+b^2+2X^2+Y^2}{4t\alpha-4t1\alpha}}\Big(-2e^{\frac{bY}{2t\alpha-2t1\alpha}}(t-t1)(b+Y)\alpha+e^{\frac{b^2+Y^2}{4t\alpha-4t1\alpha}}\sqrt{\pi}\sqrt{t-t1}\sqrt{\alpha}(Y^2+2(t-t1)\alpha)\text{Erf}\left[\frac{b-Y}{2\sqrt{t-t1}\sqrt{\alpha}}\right]\Big)+$$

$$e^{-\frac{2a^2+b^2+2X^2+2bY+Y^2}{4t\alpha-4t1\alpha}}\Big(-2(t-t1)(b-Y)\alpha+e^{\frac{(b+Y)^2}{4(t-t1)\alpha}}\sqrt{\pi}\sqrt{t-t1}\sqrt{\alpha}(Y^2+2(t-t1)\alpha)\text{Erf}\left[\frac{b+Y}{2\sqrt{t-t1}\sqrt{\alpha}}\right]\Big)\Bigg)\Bigg)$$